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# AN ANALYTICAL APPROACH TO LINEAR SYSTEMS WITH SWITCHED PARAMETERS

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# United States
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# THESIS

## AN ANALYTICAL APPROACH TO LINEAR SYSTEMS WITH SWITCHED PARAMETERS

by

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September 1971

*Approved for public release; distribution unlimited.*

# An Analytical Approach to Linear Systems with Switched Parameters

by

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# ABSTRACT

A general description of linear, switched-parameter systems is developed in terms of the mathematics of state variables. The restrictions imposed by the grouping of state variables and by the "core" states of the system upon the coefficient matrices are considered. Two procedures for determining the element values of those matrices are each illustrated with an example. A general expression for a Cost function to be used to measure system quality is developed and illustrated with two examples. Extensive recommendations for future work are made. Several examples of the utility of a Cost function minimization technique for the improvement of the step responses of some switched-parameter systems are presented.

# TABLE OF CONTENTS

I. INTRODUCTION ---------------------------------------- 7

II. MODEL CONSTRUCTION ------------------------------ 13

III. APPLICATION TO PARTICULAR LINEAR SYSTEMS -------- 25

  A. THE RESTRICTED STRUCTURE OF THE SYSTEM ------ 25

  B. THE CORE STATES ------------------------------ 27

  C. DETERMINING THE STATE EQUATIONS ------------- 32

IV. MEASURES OF SYSTEM QUALITY --------------------- 48

V. SUMMARY AND CONCLUSIONS ------------------------ 60

VI. RECOMMENDATIONS FOR FURTHER WORK ---------------- 62

APPENDIX A: APPLICATION OF COST FUNCTION MINIMIZATION ------------------------------ 67

LIST OF REFERENCES ---------------------------------- 90

INITIAL DISTRIBUTION LIST --------------------------- 91

FORM DD 1473 ---------------------------------------- 92

## LIST OF ILLUSTRATIONS

1. Servomotor Block Diagram ---------------------------- 25

2. Block Diagram of a Third Order, Switched-Parameter Control System which has a Core of States Showing Third Order Behavior Before and Second Order Behavior After Switching ------- 38

3. Time Response of the Third Order, Switched-Parameter System of the First Example to a 10 Volt Step Input ----------------------------- 41

4. Time Response of a Second Order System with Constant Parameters to a 10 Volt Step Input ------------------------------------------ 42

5. Block Diagram of the Third Order, Switched-Parameter System of the Second Example Prior to Switching ---------------------------------------- 43

6. Block Diagram of the Third Order, Switched-Parameter System of the Second Example After Switching ------------------------------------------ 44

7. Time Response of the Third Order, Switched-Parameter System of the Second Example to a 10 Volt Step Input ----------------------------- 47

8. The Cost Function of a Third Order, Switched-Parameter System as a Function of the Time to the First Switching for a 10 Volt Step Input and Where Only the Square of the Deviation of the States from the Desired Final Value is Counted as Cost ------------------------------ 54

9. The Cost Function of Figure 8 Modified to Count the Time Integral of the Square of Any Deviation from the Desired Final State During the Entire Second Control Subinterval as Cost, for System Input Steps of Various Amplitudes ---- 55

10. Same as Figure 9 Except that Time to the First Switching is Allowed to Vary in the Interval [0,5] ------------------------------------ 56

11. Replica of Figure 2 ------------------------------- 67

12. Replica of Figure 9 ------------------------------- 68

13. Replica of Figure 3 ---------------------------- 70

14. Time Response of the Third Order, Switched-Parameter System of Figure 11 to a 5 Volt Input Step ---------------------------- 71

15. Time Response of the Third Order, Switched-Parameter System of Figure 11 to a 7 Volt Input Step ---------------------------- 72

16. Time Response of the Third Order, Switched-Parameter System of Figure 11 to a 15 Volt Input Step ---------------------------- 73

17. Block Diagram of a Second Order, Switched-Parameter System Before and After Switching ---------------------------- 74

18. Cost Function of the Second Order, Switched-Parameter System of Figure 17 as a Function of the Time to the First Switching for a 10 Volt Step Input ---------------------------- 77

19. Time Response of the Second Order, Switched-Parameter System of Figure 17 after the Switching Time, $t_1$, has been Optimized --------- 78

20. Replicas of Figures 5 and 6 ---------------------------- 80

21. Replica of Figure 7 ---------------------------- 81

22. Cost Function for the Third Order, Switched-Parameter System of Figure 20, as a Function of the Time to the First Switching for a 10 Volt Step Input, where only the Square of the Deviation of the States from the Desired Final Value is Counted as Cost ---------------- 82

23. Optimum Time Response, $\hat{x}_2(t)$, of the Third Order, Switched-Parameter System of Figure 20 Obtained by Minimizing the Cost Function of Figure 22 with Respect to the Time to the First Switching ---------------------------- 84

24. Optimum Time Response, $\hat{x}_3(t)$, for the Same System as in Figure 23 ---------------------------- 85

25. Cost Function for the Third Order, Switched-Parameter System of Figure 20 as a Function of the Time to the First Switching for a 10 Volt Input Step, where the Square of Both the Deviation of the States from the Desired Endpoint Values and the Deviation of the States from the Desired Final State During the Second Subinterval are Counted as Cost ---------------- 86

26. Optimum Time Response, $\hat{x}_2(t)$, of the Third Order, Switched-Parameter System of Figure 20 Obtained by Minimizing the Cost Function of Figure 25 with Respect to the Time to the First Switching -------------- 88

27. Optimum Time Response, $\hat{x}_3(t)$, for the Same System as in Figure 26 -------------------------- 89

# I. INTRODUCTION

The first attempt to describe the behavior of a physical (real-world) system is usually based on a linear model of the system because such models are mathematically tractable and have been extensively studied. One purpose of this thesis is to develop a linear model using state variable theory for some real-world systems which may be modeled as linear systems with switched parameters. The parameters to be switched are the coefficients of the variables in the state variable representation of the system and the values the parameters take reflect the actions of various switches in the real-world system. Although the model that is developed may be applied to some kinds of electronic circuits, this thesis deals primarily with linear, switched-parameter models of control systems.

One example of the kind of real-world systems for which a model was sought is the arm positioning servomechanism in a magnetic disk-file. This servomechanism has two tasks. The first is to move the arm so that the read/write heads on the arm move from one "track" to some other in the shortest time possible. A "track" is defined as being one of the narrow concentric rings on the disk's surface which is used to store information. The second task of the servomechanism is to maintain the read/write heads (hence, the arm) in a nearly constant position with respect to the

track of interest. The arm is positioned with respect to a reference set of pre-recorded tracks which reside on a special disk.

During the transfer of the arm from one track to another, the servomechanism exhibits two modes of operation. While in the first mode, the system behaves as a "Bang-Bang" control system, undergoing first a period of maximum acceleration, followed by a period of maximum deceleration. During the time that this mode is in effect, the position of the arm is reckoned by counting tracks as they are passed. At the end of the deceleration phase of the first mode, the arm is positioned in the near vicinity of the new track. At that time the servomechanism is switched to the second mode wherein it behaves as a linear, high-gain, compensated, closed-loop position control system. This second mode is necessary in order to obtain the required tracking accuracy. The transition between modes is accomplished by switching into the system a linear error detector and a compensator, and by disabling the "Bang-Bang" circuitry.

The behavior of a system such as that just described may be modeled during each mode of operation by considering the "Bang-Bang" part and the linear position control servo part separately. However, the behavior of the system during the transition from one mode to the next is not describable by either of the two kinds of system alone. This thesis offers a solution to the problem by developing a general linear, switched-parameter model which is applicable to

these kinds of systems and which allows one to deal much more directly with the transition between modes. The model developed uses state variable concepts and is reached as the limiting case of a general, $n^{th}$ order, linear, time-varying system. The model is shown to be applicable to a restricted class of nonlinear systems.

Previous work dealing with switched parameters in linear control systems has dealt primarily with the use of discontinuous damping factors in various kinds of servomechanisms [1-4]. A different and particularly interesting application of switched-parameter techniques is that reported by Flügge-Lotz and Taylor [5]. The nonlinear system that they describe is a linear control system with multiple feedback paths and a relay switching mechanism for determining which one of the feedback paths is to be used at a particular time. The work presented in this thesis is believed to be the first attempt to develop a model applicable to the above kinds of systems.

The second purpose of this thesis was to develop a flexible and general means of measuring the "quality" of a particular switched-parameter system. As an example of what is meant by the "quality" of such a system (and as another example of what is meant by "switched-parameter system") consider the switched-parameter version of a velocity control servomechanism which is to be used to bring a shaft from a stopped condition up to a certain specified rotational velocity. The shaft is to be brought up to

speed by following a pre-planned acceleration curve until the shaft velocity is within the lock-in range of a linear phase-lock system which is to maintain the shaft at the desired velocity to within close tolerances. The inputs to the phase-lock system are a reference frequency and the output of a rotational-velocity-to-frequency transducer which is coupled to the shaft. At the instant when the rotational velocity of the shaft reaches a certain value within the lock-in range, the system is to be switched from its acceleration mode to the velocity control servo mode. In such a system, one is concerned with how closely the programmed acceleration curve is followed, how much variation in the velocity exists while the system is in the velocity control servo mode, and the extent and duration of the switching transients which may occur when the mode is shifted. The "quality" of the system is some measure of how closely the system response approaches the design goals in each case.

In order to have a single measure of the system's quality, the concept of a Cost function was taken from Optimal Control theory and applied to these systems. Besides providing one number as an indication of the quality of a switched-parameter system, the use of a Cost function permits one to utilize function minimization techniques in the design of the system. The use of such techniques was desired in order to "optimize" the performance of such systems. For instance, Cost function minimization could be applied to the velocity

servo just described in order to minimize the transient due to the switching from the acceleration mode to the velocity servo mode. Such a technique might also be applied to determine the optimum charge on one of the capacitors in the compensator which is switched into the arm positioning servomechanism in the disk-file in order to minimize the switching transient in that system.

The usefulness of the Cost function minimization technique is demonstrated in the Appendix where the step function responses of several simple switched-parameter systems are "optimized" by determining an "optimum" switching time which minimizes an applicable Cost function.

Preceding this demonstration is the development of the model in Section II which is followed, in Section III, by the consideration of some constraints the physical world imposes upon the model and then by the presentation of two methods of determining the model from the real-world system. The two methods are illustrated by application to two different examples of the type of switched-parameter systems which have acceleration modes followed by position control modes. The applicable state equations have been solved on a digital computer to illustrate the behavior of the systems.

The Cost function is developed in Section IV and illustrated with examples based on Cost versus switching time for the two examples presented in Section III. Section V is a presentation of a summary and some conclusions which

are reached based upon the work done and the examples used to illustrate it. Section VI presents several suggestions and recommendations of topics for further study.

# II. MODEL CONSTRUCTION

The starting point for the construction of a mathematical model to describe the linear, switched-parameter system was the general, linear, time-varying (LTV) system described as follows:

$$\begin{aligned} \dot{\underset{\sim}{x}}(t) &= \underset{\sim}{A}(t)\underset{\sim}{x}(t) + \underset{\sim}{B}(t)\underset{\sim}{u}(t) \\ \underset{\sim}{y}(t) &= \underset{\sim}{C}(t)\underset{\sim}{x}(t) + \underset{\sim}{D}(t)\underset{\sim}{u}(t) \end{aligned} \tag{1}$$

with initial conditions, $\underset{\sim}{x}_o = \underset{\sim}{x}(0)$, and for $t \varepsilon [0,T]$; and where

$\underset{\sim}{x}(t)$ is the n-vector of the system states,

$\dot{\underset{\sim}{x}}(t)$ is the time derivative of $\underset{\sim}{x}(t)$,

$\underset{\sim}{u}(t)$ is the m-vector of control inputs,

$\underset{\sim}{y}(t)$ is the r-vector of output variables,

$\underset{\sim}{A}(t)$ is an n x n matrix of coefficients,

$\underset{\sim}{B}(t)$ is an n x m matrix of coefficients,

$\underset{\sim}{C}(t)$ is an n x r matrix of coefficients, and

$\underset{\sim}{D}(t)$ is an r x m matrix of coefficients.

The determination of the $\underset{\sim}{A}$, $\underset{\sim}{B}$, $\underset{\sim}{C}$, and $\underset{\sim}{D}$ matrices is a straightforward procedure which is explained in many texts and which will not be considered here. In fact, the determination of the coefficient matrices from the physical system will be less of a problem than may be indicated by the time-varying nature of those matrices in (1) due to the restrictions which are soon to be placed upon them.

It is well known [6] that the time variation of the state vector which is the general solution to the LTV system in (1) is given by

$$\underset{\sim}{x}(t) = \underset{\sim}{Z}(t)\underset{\sim}{x}_o + \underset{\sim}{X}(t) \int_0^t \underset{\sim}{X}(s)^{-1}\underset{\sim}{B}(s)\underset{\sim}{u}(s)ds \qquad (2)$$

where $\underset{\sim}{X}(t)$ is an n x n matrix satisfying the following vector differential equation and where $\underset{\sim}{X}(t)^{-1}$ represents the matrix inverse of $\underset{\sim}{X}(t)$. $X(t)$ is determined by solving the system of differential equations

$$\dot{\underset{\sim}{X}}(t) = \underset{\sim}{A}(t)\underset{\sim}{X}(t)$$

over the interval $t\varepsilon[0,t]$, with initial conditions $\underset{\sim}{X}(0) = \underset{\sim}{I}$, where $\underset{\sim}{I}$ is the n x n identity matrix.

Let the control interval $[0,T]$ be divided into N subintervals of arbitrary length with the boundaries of the subintervals denoted by the sequence of times

$$\{t_o, t_1, t_2, \ldots, t_N\}$$

where $t_N - t_o = T$, and $t_i \geq t_{i-1}$ for $0 \leq i \leq N$.

Now let it be required of $\underset{\sim}{A}(t)$, $\underset{\sim}{B}(t)$, $\underset{\sim}{C}(t)$, and $\underset{\sim}{D}(t)$ that they all be matrices of constants for $t_1 + \delta \leq t \leq t_{i+1} - \delta$, for $\delta \ll \min_i(t_{i+1} - t_i)$. i.e. the coefficient matrices are constants except for a small interval about the subinterval boundaries. If $t_i + \delta$ is denoted as $t_i^+$ and $t_{i+1} - \delta$ is denoted as $t_{i+1}^-$, then the set of differential equations describing the state of the system in the subinterval $t\varepsilon[t_i^+, t_{i+1}^-]$ may be written as

$$\dot{\underset{\sim}{x}}(t) = \underset{\sim}{A}(t_i^+,t_{i+1}^-)\underset{\sim}{x}(t) + \underset{\sim}{B}(t_i^+,t_{i+1}^-)\underset{\sim}{u}(t) \qquad (3)$$

with initial conditions $\underset{\sim}{x}_o = \underset{\sim}{x}(t_i^+)$. The matrix $\underset{\sim}{A}(t_i^+,t_{i+1}^-)$ represents the constant value of the $\underset{\sim}{A}$ matrix in the particular subinterval, with a similar interpretation applicable to $\underset{\sim}{B}(t_i^+,t_{i+1}^-)$. In the next subinterval, the $\underset{\sim}{A}$ matrix value would be represented by $\underset{\sim}{A}(t_{i+1}^+,t_{i+2}^-)$, with a similar representation for the $\underset{\sim}{B}$ matrix.

Since the $\underset{\sim}{A}$ matrix in any particular subinterval will not, in general, have the same element values as the $\underset{\sim}{A}$ matrix in any other subinterval, and since the systems considered are LTV, it is necessary that the $\underset{\sim}{A}$ matrix in the transition region between two subintervals be LTV. If the $\underset{\sim}{A}$ and $\underset{\sim}{B}$ matrices in the transition region are denoted by $\underset{\sim}{A}'(t)$ and $\underset{\sim}{B}'(t)$ respectively, then the counterpart of (3) in the transition region may be written as

$$\dot{\underset{\sim}{x}}(t) = \underset{\sim}{A}'(t)\underset{\sim}{x}(t) + \underset{\sim}{B}'(t)\underset{\sim}{u}(t) \qquad (4)$$

with initial conditions $\underset{\sim}{x}_o = \underset{\sim}{x}(t_i^-)$ and valid for $t\varepsilon[t_i^-,t_i^+]$. The form of the solution to this set of differential equations is given by (2). The value of the states at $t = t_i^+$ is of particular interest since $\underset{\sim}{x}(t_i^+)$ serves as the initial condition vector for the differential equations (3) which are applicable in the following subinterval, $t\varepsilon[t_i^+,t_{i+1}^-]$. Then from (2) the expression for $x(t_i^+)$ may be written as

$$\underset{\sim}{x}(t_i^+) = \underset{\sim}{X}(t_i^+)\underset{\sim}{x}(t_i^-) + \underset{\sim}{X}(t_i^+)\int_{t_i^-}^{t_i^+} \underset{\sim}{X}(s)^{-1}\underset{\sim}{B}'(s)\underset{\sim}{u}(s)ds \qquad (5)$$

where, as before, $\underset{\sim}{X}(t)$ satisfies

$$\dot{\underset{\sim}{X}}(t) = \underset{\sim}{A}'(t)\underset{\sim}{X}(t)$$

over the interval $[t_i^-, t_i^+]$ with initial conditions $\underset{\sim}{X}(t_i^-) = \underset{\sim}{I}$ and the end point values of $\underset{\sim}{A}'(t)$ constrained such that

$$\underset{\sim}{A}'(t_i^-) = \underset{\sim}{A}(t_{i-1}, t_i)$$

$$\underset{\sim}{A}'(t_i^+) = \underset{\sim}{A}(t_i, t_{i+1}).$$

Similar requirements may be imposed upon the $\underset{\sim}{B}'$ matrix.

Since the interval $[t_i^-, t_i^+]$ is a transition period during which the system coefficient matrices $\underset{\sim}{A}$ and $\underset{\sim}{B}$ alter value, it is interesting to determine how the value of the state vector $\underset{\sim}{x}(t_i^+)$ varies as the transition period is made smaller and smaller. In particular, what is $\lim_{\delta \to 0} x(t_i^+)$? From (5) it is seen that the interval over which the solution of the $\underset{\sim}{X}(t)$ equation is required is just $[t_i^-, t_i^+]$. This interval goes to zero length as $\delta \to 0$. Hence

$$\lim_{\delta \to 0} \underset{\sim}{X}(t_i^+) = \underset{\sim}{X}(t_i^-) = \underset{\sim}{I}.$$

Consequently,

$$\lim_{\delta \to 0} \underset{\sim}{x}(t_i^+) = \lim_{\delta \to 0} \{\underset{\sim}{X}(t_i^+)\underset{\sim}{x}(t_i^-) + \underset{\sim}{X}(t_i^+) \int_{t_i^-}^{t_i^+} \underset{\sim}{X}(s)^{-1}\underset{\sim}{B}'(s)\underset{\sim}{u}(s)ds\}$$

Using the limiting values of $t_i^+$ and $t_i^-$ and $\underset{\sim}{X}(t_i^+)$, gives

$$\lim_{\delta \to 0} \underset{\sim}{x}(t_i^+) = \underset{\sim}{I}\underset{\sim}{x}(t_i^-) + \underset{\sim}{I} \int_{t_i}^{t_i} \underset{\sim}{X}(s)^{-1}\underset{\sim}{B}'(s)\underset{\sim}{u}(s)ds$$

$$= \underset{\sim}{x}(t_i^-), \qquad (6)$$

since the value of the integral taken over zero interval is zero for a finite integrand. The integrand will be finite so long as $\underset{\sim}{u}(t_i)$ is finite since $\underset{\sim}{B}'(t)$ is assumed to be

finite and since $\underset{\sim}{X}(t)^{-1} \to \underset{\sim}{I}$ as $t_i^- \to t_i^+$. Hence, if the transitions between constant coefficient matrices are assumed to be no-elapsed-time switching of the coefficient values (which is accomplished by no-elapsed-time switching of system parameters) the values of the state variables will not be affected by the discontinuous change of the coefficient matrices.

So far just the particular problem of the transition between adjacent subintervals within the overall control interval has been considered. However, the limit expressed in (6) now permits the entire problem to be treated in a straight-forward fashion. Because the differential equations (3) for each subinterval are known, the solution of (1) (with the piecewise constant limitations on $\underset{\sim}{A}$, $\underset{\sim}{B}$, $\underset{\sim}{C}$, and $\underset{\sim}{u}$) may be obtained by the sequential solution of N systems of n linear differential equations with constant coefficients. Thus, the differential equation

$$\dot{\underset{\sim}{x}}(t) = \underset{\sim}{A}(t_o,t_1)\underset{\sim}{x}(t) + \underset{\sim}{B}(t_o,t_1)\underset{\sim}{u}(t)$$

would be solved over the interval $t\varepsilon[t_o,t_1]$ with the initial conditions $\underset{\sim}{x}_o = \underset{\sim}{x}(t_o) = \underset{\sim}{x}(0)$. The value of the state vector at $t = t_1$, $\underset{\sim}{x}(t_1)$, would then be used as the initial condition for the system of differential equations

$$\dot{\underset{\sim}{x}}(t) = \underset{\sim}{A}(t_1,t_2)\underset{\sim}{x}(t) + \underset{\sim}{B}(t_1,t_2)\underset{\sim}{u}(t)$$

which would be solved over the interval $t\varepsilon[t_1,t_2]$. The final value of the state vector in this interval, $\underset{\sim}{x}(t_2)$ would be used as the initial condition vector for the next

subinterval. This process would then be repeated until the solution for the entire control interval had been obtained.

It has been assumed, to this point, that the coefficient matrices are all altered at the same instant, at each of the subinterval boundaries. However, it is clear from (6) that the limit will be unchanged even if the $\underset{\sim}{B}$ matrix is not altered or if it is altered at some time either before or after the $\underset{\sim}{A}$ matrix undergoes its transition. The case for which the $\underset{\sim}{A}$ and $\underset{\sim}{B}$ matrices undergo separate transitions is conveniently viewed as a modification of the control input, $\underset{\sim}{u}(t)$, caused by the variation of the $\underset{\sim}{B}$ matrix, applied to a system for which the A matrix elements are switched at the times $t_1, t_2, \ldots, t_{N-1}$.

The system may of course be treated as other than sequential, linear, constant systems by using some particular expression for the time variation of the $\underset{\sim}{A}'$ matrix during the transition period $[t_i^-, t_i^+]$ subject to the constraints on the endpoint values of the $\underset{\sim}{A}'$ matrix mentioned previously. It is clear, however, that if the transition period is a small fraction of the shortest natural time constant of the system, then the instantaneously switched model of the transition will describe the behavior very well. If the transition period is not small with respect to the shortest natural period of the system, then the LTV problem must be solved for the transition period in order to obtain the initial conditions for the next subinterval in which the system has constant coefficients. The speed

of present day switching devices makes the no-elapsed-time transition a practical assumption for many systems. As a result, in the material that follows it is assumed that the transitions in all of the coefficient matrices are caused by an ideal (instantaneous) switching action.

For completeness sake, it should be mentioned that the continuity of the state variables at the transition instants does not imply that the components of the output vector, $\underset{\sim}{y}$, are continuous. Indeed, this is obvious from (1), since for $\underset{\sim}{x}(t)$ and $\underset{\sim}{u}(t)$ both continuous functions of time, any discontinuity in the $\underset{\sim}{C}$ or $\underset{\sim}{D}$ matrices will appear as a discontinuity in $\underset{\sim}{y}(t)$. Let

$$\Delta \underset{\sim}{C}_i = \underset{\sim}{C}(t_i, t_{i+1}) - \underset{\sim}{C}(t_{i-1}, t_i)$$

and

$$\Delta \underset{\sim}{D}_i = \underset{\sim}{D}(t_i, t_{i+1}) - \underset{\sim}{D}(t_{i-1}, t_i)$$

be the difference matrices for the switching at $t_i$, where $1 \leq i \leq N-1$. Then the vector $\Delta \underset{\sim}{y}_i = \underset{\sim}{y}(t_i^+) - \underset{\sim}{y}(t_i^-)$ will be the vector discontinuity caused by the discontinuous changes in the $\underset{\sim}{C}$ and $\underset{\sim}{D}$ matrices. Thus,

$$\Delta \underset{\sim}{y}_i = \underset{\sim}{C}(t_i, t_{i+1})\underset{\sim}{x}(t_i^+) + \underset{\sim}{D}(t_i, t_{i+1})\underset{\sim}{u}(t_i^+)$$
$$- \underset{\sim}{C}(t_{i-1}, t_i)\underset{\sim}{x}(t_i^-) - \underset{\sim}{D}(t_{i-1}, t_i)\underset{\sim}{u}(t_i^-).$$

Having already assumed that $\underset{\sim}{x}(t)$ and $\underset{\sim}{u}(t)$ are continuous functions of time and, thus, do not change value at the switching instant, $\Delta \underset{\sim}{y}_i$ becomes

$$\Delta \underset{\sim}{y}_i = \Delta \underset{\sim}{C}_i \underset{\sim}{x}(t_i) + \Delta \underset{\sim}{D}_i \underset{\sim}{u}(t_i). \qquad (8)$$

If, however, $\underset{\sim}{u}(t)$ is allowed to change discontinuously ($\underset{\sim}{x}(t)$ is still constrained to be continuous) at the instant $t_i$, with

$$\Delta \underset{\sim}{u}_i = \underset{\sim}{u}(t_i^+) - \underset{\sim}{u}(t_i^-),$$

then, substituting for $\underset{\sim}{u}(t_i^+)$ in (7), and with $\underset{\sim}{x}(t_i^+) = \underset{\sim}{x}(t_i^-)$, gives

$$\Delta \underset{\sim}{y}_i = \Delta \underset{\sim}{C}_i \underset{\sim}{x}(t_i^+) + \underset{\sim}{D}(t_i, t_{i+1})\ \underset{\sim}{u}(t_i^-) + \Delta \underset{\sim}{u}_i - \underset{\sim}{D}(t_{i-1}, t_i)\underset{\sim}{u}(t_i^-).$$

Hence, the difference vector becomes

$$\Delta \underset{\sim}{y}_i = \Delta \underset{\sim}{C}_i \underset{\sim}{x}(t_i) + \Delta \underset{\sim}{D}_i \underset{\sim}{u}(t_i^-) + \underset{\sim}{D}(t_i, t_{i+1})\Delta \underset{\sim}{u}_i. \qquad (9)$$

The expression (9) for the discontinuity of $\underset{\sim}{y}(t)$ is perfectly general in the sense that it applies to any discontinuous change of the $\underset{\sim}{C}$ or $\underset{\sim}{D}$ matrices or the control vector $\underset{\sim}{u}(t)$ independent of any changes in either the $\underset{\sim}{A}$ or $\underset{\sim}{B}$ matrices. A change in these latter two matrices will have an indirect effect on $\underset{\sim}{y}(t)$ since their change can be expected to influence the behavior of the state vector, $\underset{\sim}{x}(t)$.

The preceding work is founded upon knowing the answers to two questions. Both of the questions exist at the interface between the analysis and the design of the types of systems under consideration. That is, one must have first solved the synthesis or design problem before the two questions may be answered and before the analysis may proceed.

The first question arises because the preceding development assumed á priori knowledge of the switching times $t_1, t_2, \ldots, t_{N-1}$. Because the manner in which these times are selected can completely alter the character of the system, the effects of the selection methods should be considered. If the N-1 switching instants are selected in some fashion prior to the beginning of the control interval, then the analysis of the system conceived as a sequence of constant, linear systems would proceed as described above. The examples presented in Sections III and IV and Appendix A are all of this type. On the other hand, the switching instants may not all be fixed prior to the beginning of the control interval, but may instead be chosen one (or two or three) at a time during the control interval, with their selection being based upon some criteria involving the current and projected state of the system. The real-world systems described in the Introduction are all examples of systems which determine the appropriate switching instant by comparing the system's current state with a predetermined state which is fixed by the system designer. These systems are intrinsically nonlinear. The nonlinearity occurs since the functional dependence of the switching instants must be displayed as

$$t_i = t_i(\underset{\sim}{x}(t),\ \underset{\sim}{u}(t), t).$$

This functional dependence of the switching instants requires that the coefficient matrices be considered, in general,

not only functions of time, but functions of the state and control vectors as well.

The functional dependence of the $\underset{\sim}{A}$ and $\underset{\sim}{B}$ matrices may be displayed as:

$$\begin{aligned} \underset{\sim}{A} &= \underset{\sim}{A}(\underset{\sim}{x}(t),\underset{\sim}{u}(t),t) \\ \underset{\sim}{B} &= \underset{\sim}{B}(\underset{\sim}{x}(t),\underset{\sim}{u}(t),t) \end{aligned} \qquad (10)$$

with similar expressions for $\underset{\sim}{C}$ and $\underset{\sim}{D}$. That the system is inherently nonlinear is evident since the character of the system's response will be, in general, dependent upon the input. However, the matrices are still constrained to be piecewise constant over the various subintervals of the control interval [0,T]. Consequently, the general functional relationships displayed above describe only formally the functional dependence of the very narrow class of systems which are considered here.

If, once the switching instant is determined, the transitions of the coefficient matrices proceed without regard for the state or control vectors, then the limiting argument used in obtaining (6) shows that the states are still continuous at the switching instant, and that the linear, piecewise constant model can still be used to give precise results for what is technically a nonlinear system. The applicability of the model to a variety of real systems is thus assured.

The second question that requires an answer before there can be a completed mathematical model probably has no "right" answer. It requires that one have determined the

best structure for the system which is to be employed. If the control interval is divided into two or more non-zero subintervals, then it is necessary to determine which sequence of coefficient matrices is the best. Alternatively, one may seek that sequence of linear, piecewise-constant-parameter control systems which would best accomplish the required task. This problem may be formalized as that of finding, for instance, the $\underset{\sim}{A}$ matrix of a particular system; i.e. find the function, $\underset{\sim}{f}$, such that

$$\underset{\sim}{A}(t_i,t_{i+1}) = \underset{\sim}{f}(\underset{\sim}{x}(t),\underset{\sim}{u}(t),t,\underset{\sim}{A}(t_{i-1},t_i))$$

for $1 \leq i \leq N-1$, will cause the resultant system to have the desired characteristics. This problem remains to be solved. The best that may be expected, at present, is that some sort of heuristic approach which is aided by the physical constraints that a particular application imposes will lead to a system which gives acceptable performance.

From the preceding it has been seen that linear, switched-parameter systems are easily handled by the mathematics of the state variable theory. It is apparent, however, that the set of coupled differential equations that describes one of these systems is probably more amenable to being solved on a digital computer than to being expressed in a mathematically closed form. The closed form expression is possible in principle but would become increasingly clumsy as the system order and the number of subintervals increased.

Another possibility is that hybrid computer techniques might be used to obtain a solution to the state equations. In applications that do not require a great deal of accuracy in the solution, such an approach may prove useful; however, the insensitivity of the hybrid computer to small but significant changes in a parameter value or the length of a control subinterval may not allow the use of these techniques.

# III. APPLICATION TO PARTICULAR LINEAR SYSTEMS

## A. THE RESTRICTED STRUCTURE OF THE SYSTEM

The general switched-parameter system which was the concern of the previous section will seldom be encountered in its full generality. One reason for this is that physical systems exhibit a part of their reality as restrictions on the forms that the coefficient matrices of the state variable representation are free to take. As an example of this sort of restriction may be cited that shown by the classical linear description of a servomotor. The transfer function is shown with a block diagram of the servomotor in Figure 1.

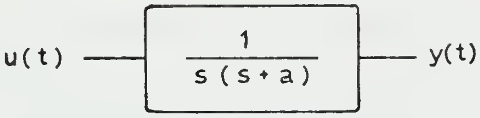


Figure 1. Servomotor Block Diagram.

The second order differential equation which describes the behavior of this servomotor is

$$\ddot{y}(t) + a\dot{y}(t) = ku(t), \tag{11}$$

where the dots indicate the time derivatives and a and k are constants.

In putting (11) into state variable form it is common practice to choose as the states of the system the function

$y(t)$ and its time derivative $\dot{y}(t)$. So let

$$x_1(t) = y(t)$$

and

$$x_2(t) = \dot{y}(t)$$

be the system states. Then, rewriting (11) in state variable form:

$$\begin{aligned} \dot{x}_1(t) &= x_2(t) \\ \dot{x}_2(t) &= -ax_2(t) + ku(t), \end{aligned} \qquad (12)$$

which may be put into matrix form as

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & -a \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ k \end{bmatrix} \begin{bmatrix} u(t) \end{bmatrix}, \qquad (13)$$

or

$$\dot{\underset{\sim}{x}}(t) = \underset{\sim}{A}\underset{\sim}{x}(t) + \underset{\sim}{B}\underset{\sim}{u}(t).$$

The point of this elementary exercise has been to show that the forms of the $\underset{\sim}{A}$ and the $\underset{\sim}{B}$ matrices are restricted by the particular device and by the quantities selected as the state variables. In (13) the elements of the first column of the $\underset{\sim}{A}$ matrix will always be zero for this particular choice of states to describe this particular device. Additionally, the element of the $\underset{\sim}{B}$ matrix that is presently zero, will always be zero under these conditions. Indeed, one may say that for an $n^{th}$ order system which has this servomotor in it and which uses $y(t)$ and $\dot{y}(t)$ as the two states that describe the behavior of the servomotor (and

n-2 other states to describe the rest of the system), if $x_i(t)$ corresponds to $y(t)$ and $x_j(t)$ corresponds to $\dot{y}(t)$ $(i \neq j)$, then

$$\dot{x}_i(t) = x_j(t), \tag{14}$$

i.e. the only non-zero term in the $i^{th}$ row of the system's $\underset{\sim}{A}$ and $\underset{\sim}{B}$ matrices will be $a_{ij} = 1$. The restriction on the elements of the two coefficient matrices result because the servomotor is described by a second order differential equation which, when expressed as coupled first order differential equations will always result in (14) as the first order differential equation for $x_i(t)$ $(= y(t))$. Furthermore, the two states $x_i(t)$ and $x_j(t)$, or their equivalents, will always be present and always be paired as in (14) in any system in which the servomotor is used (still assuming the basic $y(t)$ and $\dot{y}(t)$ choice of states).

## B. THE CORE STATES

The concept of grouping the system's state variables as occurred with $x_i(t)$ and $x_j(t)$ above forms the basis for the following discussion of switched-parameter control systems. No matter what the system's order may be, there will always be a "core" of states which one is trying to control. For instance, in a shaft positioning servomechanism, one seeks to control the states of the servomotor which drives the shaft and those states (shaft position and shaft rotational velocity) become the "core" states of the system. It may be that various compensation networks have been employed in

the system to improve its response. The state variables needed to describe the behavior of these networks, taken together with the "core" states, make up the n states necessary to describe the system.

The essential point is that one seeks to control the values of only a few of the total number of the system's states. This does not mean that the remaining states are not to be controlled in some sense. It does mean that one is less concerned with their endpoint values than with the effect they may have on the variation of the "core" states.

One is led naturally, then, to questions concerning the order of the variation of a particular state (or set of states) within a system. If the system can be represented by a set of first order differential equations as

$$\left[\begin{array}{c} \dot{\underset{\sim}{x}}_o(t) \\ \hline \dot{\underset{\sim}{x}}_1(t) \end{array}\right] = \left[\begin{array}{c|c} \underset{\sim}{A}_o & \underset{\sim}{A}'_o \\ \hline \underset{\sim}{0} & \underset{\sim}{A}_1 \end{array}\right] \left[\begin{array}{c} \underset{\sim}{x}_o(t) \\ \hline \underset{\sim}{x}_1(t) \end{array}\right] + \left[\begin{array}{c} \underset{\sim}{B}_o \\ \hline \underset{\sim}{B}_1 \end{array}\right] \left[\underset{\sim}{u}(t)\right], \qquad (15)$$

where, for an $n^{th}$ order system, $\underset{\sim}{A}_o$ is an r x r matrix, $\underset{\sim}{A}'_o$ is an r x (n-r) matrix, $\underset{\sim}{A}_1$ is an (n-r) x (n-r) matrix, and $\underset{\sim}{0}$ is a (n-r) x r null matrix, then, one may say that the states comprising the state vector $\underset{\sim}{x}_1(t)$ will exhibit variations characteristic of an $(n-r)^{th}$ order system since they are totally decoupled from the r states represented by the $\underset{\sim}{x}_o(t)$ state vector. The states of the $\underset{\sim}{x}_o(t)$ state vector will exhibit the variations characteristic of an $r^{th}$ order system,

$$\dot{\underset{\sim}{x}}_o(t) = \underset{\sim}{A}_o\underset{\sim}{x}_o(t) + \underset{\sim}{A}'_o\underset{\sim}{x}_1(t) + \underset{\sim}{B}_o\underset{\sim}{u}(t),$$

where the last two terms represent the forcing function for the inhomogeneous system.

The linear, switched-parameter systems which are represented in a particular time interval by (15) are a subset of the more general class of switched-parameter system described by

$$\left[\begin{array}{c}\dot{\underset{\sim}{x}}_o(t)\\ \hline \dot{\underset{\sim}{x}}_1(t)\end{array}\right] = \left[\begin{array}{c|c}\underset{\sim}{A}_o & \underset{\sim}{A}'_o\\ \hline \underset{\sim}{A}'_1 & \underset{\sim}{A}_1\end{array}\right]\left[\begin{array}{c}\underset{\sim}{x}_o(t)\\ \hline \underset{\sim}{x}_1(t)\end{array}\right] + \left[\begin{array}{c}\underset{\sim}{B}_o\\ \hline \underset{\sim}{B}_1\end{array}\right]\left[\underset{\sim}{u}(t)\right] \tag{16}$$

which is the same as (15) except that the (n-r) x r null matrix is generalized to the matrix $\underset{\sim}{A}'_1$. The $\underset{\sim}{x}_1(t)$ state vector for this case (as for (15) when that equation is interpreted as describing a switched-parameter system) represents those of the system states which are considered the "core" states.

The switched-parameter system to be considered will be described by a sequence of N linear, constant-parameter systems during N successive subintervals of the overall control interval which is taken as $[0,T]$. During the $i^{th}$ subinterval the system is described by (16) with the initial conditions, $\underset{\sim}{x}_o = x(t_i^-)$, and for $t \epsilon [t_i, t_{i+1}]$. With $\underset{\sim}{A}'_1$ and $\underset{\sim}{A}'_o$ taken as non-null matrices and assuming that $\underset{\sim}{A}$ is rank n (non-degenerate system), the "core" state vector, $\underset{\sim}{x}_1(t)$ may have $n^{th}$ order behavior. It will be assumed that it does. The partitioning of the $\underset{\sim}{A}$ matrix in (16) serves to delimit the "core" states. The system's differential equations have been so ordered that such a partitioning is possible. Now, assume that in the interval $[t_{i+1}, t_{i+2}]$ the behavior of the

"core" states is to be less than $n^{th}$ order. Such behavior could be caused by forcing the $\underset{\sim}{A}_1'$ matrix to become $[\underset{\sim}{0} \,\vdots\, \underset{\sim}{A}_1'']$ in the $(i+1)^{st}$ subinterval. This may require that the equations of the states not in the "core" be rearranged.

The discussion will proceed as well if the $\underset{\sim}{A}_1'$ matrix is forced to become the null matrix. Thus, the behavior of the "core" states will be altered from $n^{th}$ order to $(n-r)^{th}$ order at $t = t_{i+1}$. It is understood that the "forcing" of the $\underset{\sim}{A}_1'$ matrix is to be done by switching the values of the system's parameters. The next linear system to follow that described by (16), then, will be the system (15) with initial conditions $\underset{\sim}{x}_o = \underset{\sim}{x}(t_{i+1}^-)$ and valid for $t \varepsilon [t_{i+1}, t_{i+2}]$. The initial conditions for (15) may be viewed as the initial conditions for the first subinterval ($\underset{\sim}{x}(0)$) transformed by the intervening sequence of linear systems to their present value, $\underset{\sim}{x}(t_{i+1}^-)$. The effect of the preceding systems is, thus, apparent. It may be noted that if knowledge of the variations of $\underset{\sim}{x}_o(t)$ is not required for further work, the entire system of n differential equations need not be solved. One may solve for this special case just the equations

$$\dot{\underset{\sim}{x}}_1(t) = \underset{\sim}{A}_1 \underset{\sim}{x}_1(t) + \underset{\sim}{B}_1 \underset{\sim}{u}(t), \qquad (17)$$

with the initial conditions being the appropriate subset of the initial conditions for (15).

Since the order of the behavior of the "core" state vector is (n-r), which is assumed to be the number of the "core" states, the system will be restricted in the changes

that may be made in it at the end of the current subinterval. That is, the character of the behavior of the "core" states may remain of the same order, (n-r), or the order may be increased (up to $n^{th}$ order) by suitably altering the $\underset{\sim}{A}'_1$ and the $\underset{\sim}{A}'_o$ matrices to couple additional differential equations to the equations of the "core" states. If the order is to be increased, the values of those states in the $\underset{\sim}{x}_o(t)$ vector which are to be coupled to the $\underset{\sim}{x}_1(t)$ state vector at the next switching instant will become quite important since their values at the endpoint of the current subinterval will be used as the initial conditions for the set of coupled equations which include the "core" state equations in the next subinterval. Hence, one will be solving (17) augmented by some additional states, $\underset{\sim}{x}'_o(t)$, where the coupling is accomplished by the equivalent of the $\underset{\sim}{A}_1''$ matrix mentioned earlier as the non-null part of the $\underset{\sim}{A}'_1$ matrix; i.e. one must solve

$$\dot{\underset{\sim}{x}}'_1(t) = \underset{\sim}{A}_1\underset{\sim}{x}_1(t) + \underset{\sim}{A}''_1\underset{\sim}{x}'_o(t) + \underset{\sim}{B}'\underset{\sim}{u}(t) \qquad (18)$$

where the initial conditions are given by $\underset{\sim}{x}_1(t^-_{i+1})$ and $\underset{\sim}{x}'_o(t^-_{i+1})$ for the subinterval $t\varepsilon[t_{i+1},t_{i+2}]$, and where $\underset{\sim}{B}'\underset{\sim}{u}(t)$ represents the appropriate forcing function terms from (15).

It may be noted in passing that the initial conditions represented by $\underset{\sim}{x}'_o(t^-_{i+1})$ may be controlled to some extent by the system designer during the subinterval in which they are decoupled from the "core" states. This flexibility is allowed since the vector of states, $\underset{\sim}{x}_o(t)$, in (15) may be

treated as an independent system so long as none of its member states are coupled to the "core" state vector, $x_1(t)$. The improvements in the response of a switched-parameter system that such flexibility may permit remain to be determined. The values of $\underset{\sim}{x}'_o(0)$ (for the case in which low order behavior of the "core" states is desired during the first subinterval) are a particularly important example of the flexibility that is available.

Once the linear model of the system has been placed in the form of (15), and the elements of the A matrix that are to change and the values that they are to take after each change have been established for each segment of the control interval, then the solution of the system's differential equations may proceed straightforwardly.

## C. DETERMINING THE STATE EQUATIONS

It is obvious from the foregoing material that the description of switched-parameter systems is most easily accomplished using state variable concepts. Furthermore, if the state variables are restricted to those which have an immediate physical significance (e.g. capacitor voltages, inductor currents, shaft rotational velocities, and the like) the conceptual understanding of the switchings of the system's parameters and the resultant behavior of the system's states is enhanced since one can readily see how much of one particular state is being coupled to some other. The order of the system and the values of many of the system parameters will have been fixed during the design of the

system. Usually the fixed parameters will be fixed because of the inherent physical characteristics of the system's components; although, some parameters may be fixed by "designer's choice." Those parameters that do not have fixed values during the entire control interval for one of the above reasons are those that are to be switched in value from subinterval to subinterval.

Two procedures present themselves as possible methods for translating the physical system with switched-parameters into the state equations of (16). The first procedure would have one determine the coefficient matrices ($\underset{\sim}{A}$, $\underset{\sim}{B}$, $\underset{\sim}{C}$, and $\underset{\sim}{D}$) assuming a maximum amount of coupling between the state equations; that is, utilize individual multiplicative switching functions (having a value 0 to 1), assume all of the switching functions are non-zero, and determine a general set of coefficient matrices. Then, by allowing the appropriate switching functions to take their zero values in the proper subintervals, one may read the element values of the coefficient matrices which are appropos of that subinterval. This procedure requires that one manipulate the differential equations for an $n^{th}$ order system (assuming that the maximum order of the system is n), complicated by however many switching functions are required. For the case of N subintervals, there would be N-1 individual switching functions necessary.

The second procedure asks that one proceed on a subinterval-by-subinterval basis. Beginning in the first

subinterval, the state equations of those states which are coupled to the "core" states are written down. Next, the equations for any other states which are not coupled to the "core" states but are coupled to a forcing function or other states not in the "core" (but are not isolated) are written down, leaving only the isolated states unaccounted for. One then considers the system as it is to be after the first switching instant, identifies the isolated states which have been coupled to the "core" states, and adds rows and columns to the appropriate system coefficient matrices in order to take these "new" states into account. The elements of the coefficient matrices that correspond to the "old" states are changed appropriately so as to take account of the coupling or the uncoupling of those states. One continues in this fashion, adding rows and columns (adding differential equations) to the coefficient matrices until they have attained their maximum dimensions, after which time the coefficient matrices are determined simply by altering the appropriate element values for each subinterval. Care must be taken to ascertain the initial value of any "new" states when they are switched into the system from their previous isolation the first time.

There are some pitfalls to be avoided in both of these procedures. For the most part, the pitfalls are related to the identification of the states of the system. In switching parameters, it may occur that one obtains the equivalent of a circuit in which there is a loop containing all capacitors,

or a node having only inductors connected to it. Such a system is degenerate in that not all of the "states" represented by the capacitor voltages, or the inductor currents, are independent. This will require that the $\underset{\sim}{A}$ matrix of the system in that subinterval have rank one less (assuming only one degenerate state) than it would have if the system were not degenerate. The system of equations may still be solved but methods that require inversion of the $\underset{\sim}{A}$ matrix must be used with care. The use of iterative differential equation solvers such as the Runge-Kutta and the Hamming's Modified Predictor-Corrector methods will provide time response solutions to the state equations with a minimum of concern about the values of the various coefficient matrices.

Another type of pitfall may be encountered when using the first procedure mentioned above. The problem occurs when allowing a switching function to take a zero value causes an element of a coefficient matrix to become infinite. This is probably the result of dividing by a switching function which was being treated as a symbolic, non-zero quantity during some algebraic manipulations. The problem is best cured by guarding against such manipulations.

One other point needs to be mentioned. The system, in each sub-interval, appears as one or more sets of coupled, linear differential equations. In many cases the system being modeled will be a classical, linear control system with a single input and a single output. Such systems are

well described using transfer functions, pole-zero plots, and the other well-known methods applicable to classical systems. Unfortunately, these classical analysis techniques are not well suited to the description of switched parameter systems even in the single-input, single-output case since the identity of the individual states is easily lost. This is not so important during the period that the system is in one particular subinterval, but it is very important at the ends of the subinterval because the values of the individual states must be known in order to determine the initial conditions for the subsequent subinterval. For instance, if in one subinterval a system has a transfer function, T(s) which changes as shown:

$$T(s) = \frac{1}{s^3} \text{ for } t \leq t_i,$$

and

$$T(s) = \frac{1}{s^2(s+a)} \text{ for } t > t_i,$$

the behavior of the system's output will depend on the structure of the second system. If the system in the subinterval prior to $t_i$ is structured as three individual 1/s terms, then the system which results from the switching of

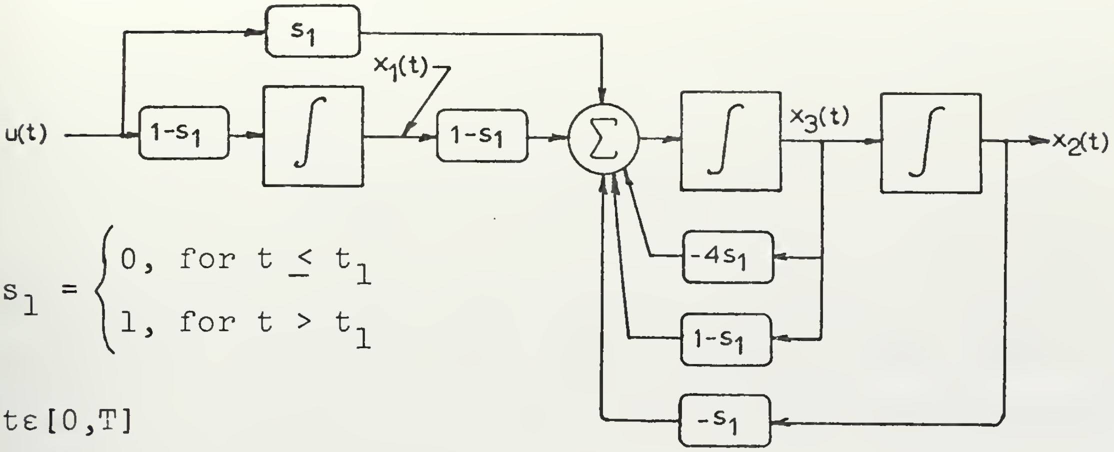
$$\left(\frac{1}{s}\right)\left(\frac{1}{s}\right)\left(\frac{1}{s}\right) \rightarrow \left(\frac{1}{s+a}\right)\left(\frac{1}{s}\right)\left(\frac{1}{s}\right)$$

will have a different behavior than if the switching goes as

$$\left(\frac{1}{s}\right)\left(\frac{1}{s}\right)\left(\frac{1}{s}\right) \rightarrow \left(\frac{1}{s}\right)\left(\frac{1}{s}\right)\left(\frac{1}{s+a}\right).$$

This will be so because the initial condition on the 1/(s+a) term will vary with its position in the system and thus cause the transient response term of the system output to vary. Since one must look at the individual state variables in order to determine the structure of the system in two adjacent subintervals, it is convenient to describe the system only in terms of the state variables. Of course, it is still possible to consider the transfer function for a particular subinterval where that is possible and where it may lend additional insight into the behavior of the system in the subinterval. This will usually be advantageous when the effect of the parameter switching will be to bring in or to isolate an entire group of states such as those describing the servomotor discussed at the beginning of this section.

A simple example will be worked for each procedure to illustrate its use. The two examples have been selected for mathematical simplicity and as illustrations of systems that show improved step response over purely linear systems. Both examples are third order systems with only one switching of the parameters considered. Single-input, single-output systems were used with the output being one of the states in order to simplify matters even further and to remove the necessity for considering the $\underset{\sim}{C}$ and $\underset{\sim}{D}$ matrices in the examples. Because of the diagrammatic simplicity, the systems are shown in a block diagram form in the time domain with each of the states represented by an integrator.



$$s_1 = \begin{cases} 0, & \text{for } t \le t_1 \\ 1, & \text{for } t > t_1 \end{cases}$$

$t \varepsilon [0,T]$

Figure 2. Block Diagram of a Third Order, Switched-Parameter Control System which has a Core of States Showing Third Order Behavior Before and Second Order Behavior After Switching.

Figure 2 shows the system that is used to illustrate application of the first procedure. The symbol $s_1$ represents the switching function and takes on the two values indicated. The identity of the states, as shown, is arbitrary provided only that they remain distinct. The first procedure requires that all of the switching functions be considered non-zero. This is easily done by treating the functions (in this case, only $s_1$) symbolically while writing down the n coupled differential equations that describe the system. Thus for the system of Figure 2, one writes

$$\dot{x}_1(t) = (1-s_1)u(t)$$

$$\dot{x}_2(t) = x_3(t)$$

$$\dot{x}_3(t) = s_1u(t) + (1-s_1)x_1(t) + (1-s_1)x_3(t)$$
$$- 4s_1x_3(t) - s_1x_2(t)$$

or

$$\underset{\sim}{\dot{x}}(t) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ (1-s_1) & -s_1 & (1-s_1)-4s_1 \end{bmatrix} \underset{\sim}{x}(t) + \begin{bmatrix} (1-s_1) \\ 0 \\ s_1 \end{bmatrix} \underset{\sim}{u}(t)$$

For $s_1$ equal to zero in the first subinterval and equal to one in the second, and with the switching taking place at $t' = t_1$, the systems on either side of the switching point are readily seen to be

$$\underset{\sim}{\dot{x}}(t) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix} \underset{\sim}{x}(t) + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \underset{\sim}{u}(t), \text{ for } t \varepsilon [0, t_1] \qquad (19)$$

and

$$\underset{\sim}{\dot{x}}(t) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & -4 \end{bmatrix} \underset{\sim}{x}(t) + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \underset{\sim}{u}(t), \text{ for } t \varepsilon [t_1, T] \qquad (20)$$

where the element values of the $\underset{\sim}{A}$ and $\underset{\sim}{B}$ matrices have been developed explicitly. For this simple case the procedure works well. However, when the system complexity increases and there are more than several switching functions to consider, then the expressions required for the individual elements of the various matrices may become unduly unwieldly. The second procedure should then be used in preference to this one.

The two sets of equations (19) and (20) were solved on a digital computer by means of an iterative routine that uses Hamming's Modified Predictor-Corrector method for

solving sets of linear differential equations. The method utilizes a fourth order Runge-Kutta procedure to generate four initial points, and then proceeds, using the preceding four points in an iterative procedure to generate an estimate of the next point, and, by extension, the solution to the set of equations. For further information on these methods see [7, 8]. Figure 3 shows the time response of the states of the system for two different values of the switching time, $t_1$. The input, u(t) was a 10 volt step function in both cases. The length of the control interval was 10 seconds. The initial conditions of all the states were taken to be zero. The switching time for Figure 3(a) was $t_1 = 1.45$ seconds, and for Figure 3(b), $t_1 = 1.349$ seconds.

Figure 4 shows the time response of the states of the second order system with constant parameters,

$$\dot{\underset{\sim}{x}}(t) = \begin{bmatrix} 0 & 1 \\ -1 & -4 \end{bmatrix} \underset{\sim}{x}(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} [u(t)],$$

to a 10 volt step input. Zero initial conditions and a 10 second control interval apply also to this system. Figure 4 is intended to serve as a reference for comparison with the time responses of the other systems. As may be seen from the comparison of Figures 3 and 4, the switched-parameter system shows a marked improvement in step response.

The comparison of these two different systems is justified since the switched-parameter systems may be viewed as being the same second order system with means for modifying its initial conditions. Thus, one is comparing the response

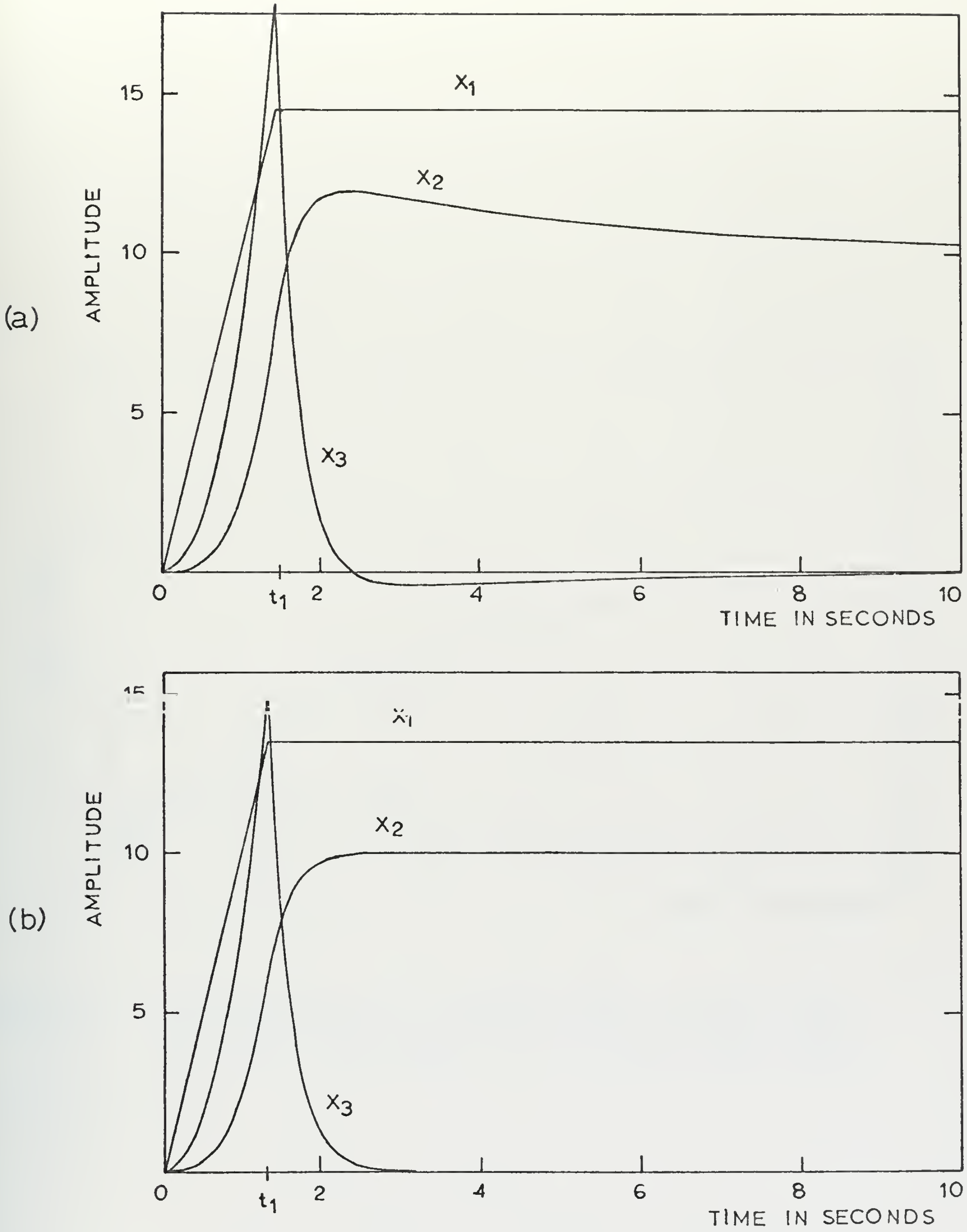


Figure 3. Time Response of the Third Order, Switched-Parameter System of the First Example to a 10 Volt Step Input. (a) Switching Time, $t_1$ = 1.45 Seconds. (b) $t_1$ = 1.349 seconds.

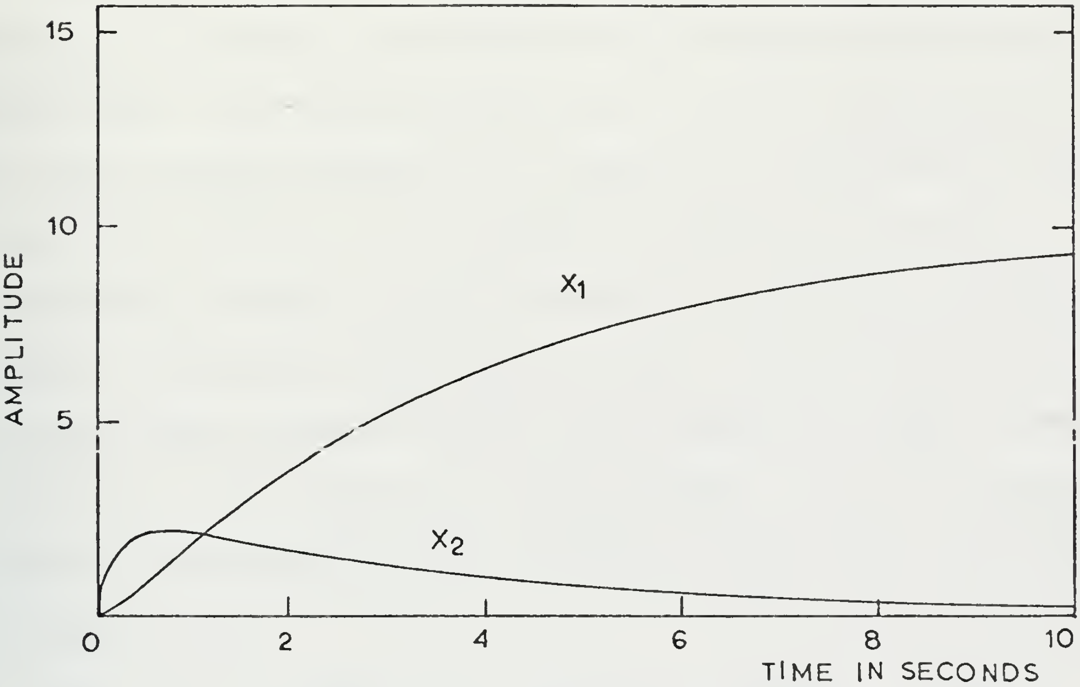


Figure 4. Time Response of a Second Order System with Constant Parameters to a 10 Volt Step Input.

of the constant-parameter, second order system when started from zero initial conditions to the response of the same system during the time its initial conditions are being "pre-charged" by the unstable third order system (in the first subinterval) and after it is released as a second order system (in the second subinterval).

The choice of switching time, $t_1$, for Figure 3(b) was decided by repeated trial-and-error solutions of the state equations. The value obtained, $t_1 = 1.349$ seconds, gave the best-looking response. An increase or decrease in this value by as much as 0.001 seconds produced a visually detectable increase in the overshoot or undershoot respectively. A similar sensitivity to the value of the switching time has been noticed in all of the examples worked. The effects of such sensitivity may preclude the use of hybrid computer techniques in the solution of the state equations of such systems.

The second example will be equally simple. Figure 5 shows the third order system in the first subinterval. As before, the states have been individually identified and represented by integrator blocks.

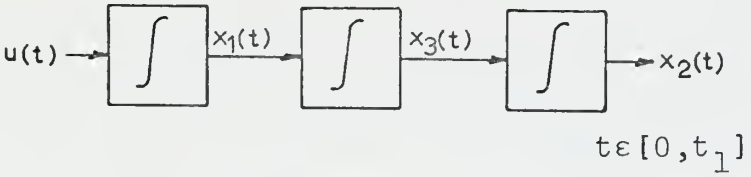


Figure 5. Block Diagram of the Third Order, Switched-Parameter System of the Second Example Prior to Switching.

The system differential equations in this subinterval are

$$\underset{\sim}{\dot{x}}(t) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \underset{\sim}{x}(t) + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} u(t) \end{bmatrix}, \qquad (21)$$

$$t \varepsilon [0, t_1].$$

It is desired that the system take the configuration shown in Figure 6 in the subinterval $t \varepsilon [t_1, T]$. The changes to be made can be systematically determined by examining the differential equation of each state in turn, as they are given in (21).

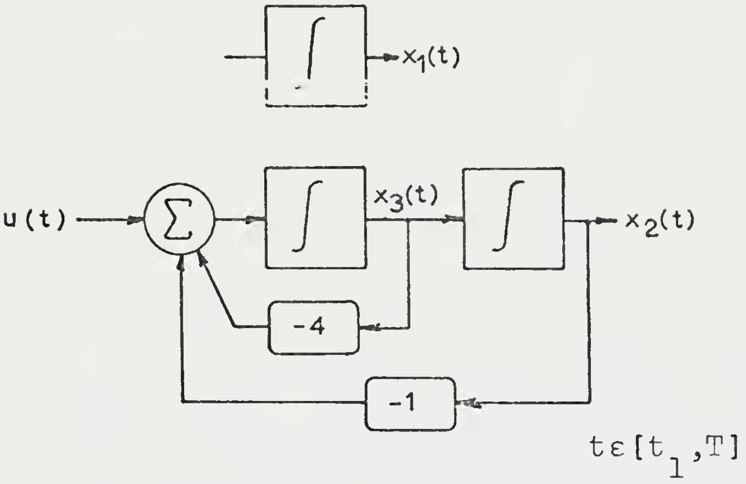


Figure 6. Block Diagram of the Third Order Switched-Parameter System of the Second Example After Switching.

Going from top to bottom of the state vector, examining $x_1(t)$, $x_2(t)$, and $x_3(t)$ in turn, it is apparent that the new set of state equations becomes

$$\underset{\sim}{\dot{x}}(t) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & -4 \end{bmatrix} \underset{\sim}{x}(t) + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \begin{bmatrix} u(t) \end{bmatrix}, \text{ for } t\varepsilon[t_1, T]. \qquad (22)$$

If an isolated fourth state had been encountered in the second subinterval, and if the coupling between the states $x_1(t)$, $x_2(t)$ and $x_3(t)$ was as shown in (22), then the fourth order system of this subinterval would be formed as

$$\left[\begin{array}{c} \underset{\sim}{\dot{x}}(t) \\ \hline \underset{\sim}{\dot{x}}_4(t) \end{array}\right] = \left[\begin{array}{ccc|c} 0 & 0 & 0 & \\ 0 & 0 & 1 & \underset{\sim}{A}'_o \\ 0 & -1 & -4 & \\ \hline & \underset{\sim}{A}'_1 & & \underset{\sim}{A}_1 \end{array}\right] \left[\begin{array}{c} \underset{\sim}{x}(t) \\ \hline x_4(t) \end{array}\right] + \left[\begin{array}{c} 0 \\ 0 \\ 1 \\ \hline \underset{\sim}{B}_1 \end{array}\right] \begin{bmatrix} u(t) \end{bmatrix} \qquad (23)$$

$$t\varepsilon[t_1, T]$$

where the elements of $\underset{\sim}{A}'_1$, $\underset{\sim}{A}_1$, $\underset{\sim}{A}'_o$, and $\underset{\sim}{B}_1$ would be determined from the coupling between $x_4(t)$ and the three prior states. It should be well noted, that since this is the first time that $x_4(t)$ has been used in the system (else it would already be included in $\underset{\sim}{x}(t)$), that particular attention must be paid to the value of $x_4(t_1)$, which is the initial condition of $x_4(t)$, for the new subinterval.

Depending upon the method used to solve the set of differential equations (22) or (23), the chore of programming a digital computer for the task of solving those equations may be eased by always working with an $n^{th}$ order system, even when there are isolated states that are being "ignored." It would then be necessary only to alter the

element values in the coefficient matrices and not necessary to change their dimensions during the course of the problem solution. The isolated states are conveniently allowed for even if not individually identified, by adding the equivalent of $\underset{\sim}{A}'_0$, $\underset{\sim}{A}_1$, $\underset{\sim}{A}'_1$, and $\underset{\sim}{B}_1$ as null matrices. If there is some trouble encountered in determining the maximum system order, the lower order system's equations may be written down and then adjoined to the appropriate null matrices, when the maximum system order is known.

As for the first example system, this system, as described by (21) and (22) was solved on the digital computer. The trajectories of the states are shown in Figures 7(a) and (b) for a control interval of 10 seconds and a 10 volt step function input. Figure 7(b) is for a switching time, $t_1 = 1.349$ seconds, and 7(a) is for $t_1 = 1.699$ seconds. Figure 7(a), when compared with Figure 3(a), shows that the response of this system has a slightly longer rise time than the first system for about the same amount of overshoot. The response of this system is also markedly better than that of the plain second order system.

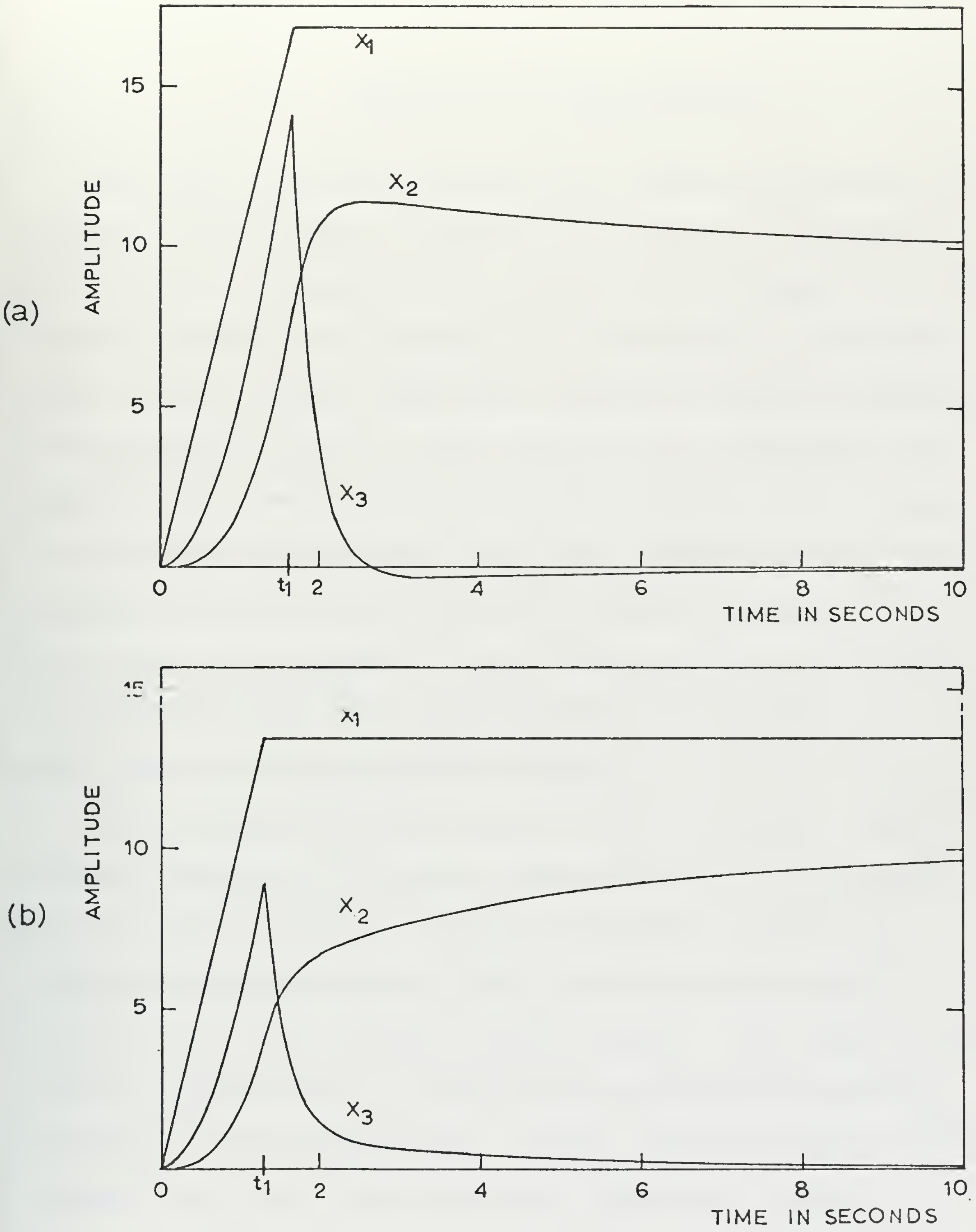


Figure 7. Time Response of the Third Order, Switched-Parameter System of the Second Example to a 10 Volt Step Input. (a) Switching Time, $t_1$ = 1.699 Seconds. (b) $t_1$ = 1.349 seconds.

## IV. MEASURES OF SYSTEM QUALITY

The various methods available for taking the measure of a system (or its model) have the common goal of providing the analyst or designer with a quantitative estimate of how well the system may be expected to accomplish its designed tasks. Linear system models are commonly tested by applying a known input (usually a step function or a sinusoid) and comparing the output with the desired output. In the case of the step function input, one might examine the rise time, overshoot, and settling time of the output response. As may be seen from Figures 3 and 7, there seems to be justification for using those measures of system quality to grade some switched-parameter systems.

Unfortunately, the usefulness of the classical system response measures is somewhat degraded since it is usually not possible to make the direct correlation between the classical response measure and a system parameter that is possible with the classical linear system. The restriction results from what may be considered an inherent ambiguity in linear, switched-parameter systems. The ambiguity arises because there may be more than one sequence of linear, constant-parameter systems that would have the same output for a given input. This is another manifestation of the intrinsic nonlinearity of switched-parameter systems. Still, for those switched-parameter systems that are designed to

replace a classical linear system, the classical response measures may well provide a useful means of grading the systems.

There are cases, however, for which the classical measures of system performance are not adequate. As one example may be cited the Linear Regulator problem of Optimal Control theory. A Cost function approach to the measurement of the quality of linear, switched-parameter systems has several features to recommend it. The possibility of having several subintervals in which such systems may have greatly varied behavior and in each of which the designer may have separate goals, strongly recommends a flexible measure of system quality. In addition, a Cost function may be useful in providing a measure of several variables in the system at the same time. For instance, one might construct a Cost function for a system whose value would be the combined measure of the system's settling time and the integral of the error between the output and the input squared (assuming a system for which such things make sense.) Another reason for desiring to use Cost functions with switched-parameter systems is that such an approach lends itself to some sort of system optimization accomplished through minimization of the Cost function. Several examples of Cost function minimization are presented in Appendix A as a demonstration of the utility of the method. The examples are limited to the determination of the optimum switching time for some simple systems but they illustrate a method that might profitably be employed.

Against this potential flexibility must be balanced the loss of detail that is inherent in the use of any sort of Cost function. The loss of detail arises, of course, since the Cost function is the weighted sum of several different quantities and there is always the possibility that an increase in one quantity was offset by a decrease in another. Ambiguity in the value of the Cost function is possible when the changes in the different quantities offset each other so that the function value remains unchanged.

A general Cost function for switched-parameter systems may be written as

$$J_T = \underset{\sim}{Q}^T \underset{\sim}{J}_S + RT_c \qquad (24)$$

where

$\underset{\sim}{Q} \equiv [Q_1, Q_2, .., Q_n]^T$ and $Q_i \geq 0$ for $i = 1, N$,

$R \geq 0$,

are weighting factors for the Cost vector, $\underset{\sim}{J}_S$, and the length of the control interval, $T_c$, respectively. The vector

$$\underset{\sim}{J}_S \equiv [J_1, J_2, \ldots, J_N]^T$$

the length of the control interval is defined as

$$T_c = \sum_{i=1}^{N} (t_i - t_{i-1}),$$

for the switched-parameter system given by

$$\dot{\underset{\sim}{x}}(t) = \underset{\sim}{A}(t)\underset{\sim}{x}(t) + \underset{\sim}{B}(t)\underset{\sim}{u}(t),$$

and valid for t in the interval $[0, T_c]$.

In the $i^{th}$ subinterval,

$$J_1 = \underset{\sim}{w}(t_{i+1})^T \underset{\sim}{H}_i \underset{\sim}{w}(t_{i+1}) + \int_{t_i}^{t_{i+1}} [\underset{\sim}{v}(s)^T \underset{\sim}{M}_i \underset{\sim}{v}(s) + \underset{\sim}{u}(s)^T \underset{\sim}{N}_i \underset{\sim}{u}(s)]ds \tag{25}$$

$$\underset{\sim}{w}(t_{i+1}) = \underset{\sim}{x}(t_{i+1}) - \underset{\sim}{x}_D(t_{i+1})$$

$$\underset{\sim}{v}(t) = \underset{\sim}{x}(t) - \underset{\sim}{r}_i(t),$$

where $\underset{\sim}{H}_i$, $\underset{\sim}{M}_i$, and $\underset{\sim}{N}_i$ are n x n, n x n, and m x m positive semi-definite diagonal matrices of weighting factors, respectively. The vector, $\underset{\sim}{r}_i(t)$, is the desired trajectory for $\underset{\sim}{x}(t)$ in the subinterval $t \varepsilon [t_i, t_{i+1}]$, and $\underset{\sim}{x}_D(t_{i+1})$ is the desired value of $\underset{\sim}{x}(t)$ at $t = t_{i+1}$. An equation analgous to (25) is applicable to each of the N subintervals.

Which of the weighting factors are to be non-zero, and what the values are to be is left to the system analyst for decision since the weighting factors should be chosen to provide a measure of those things that are of interest in a particular system when it is used for a particular purpose.

Although a general statement of the useful forms of the Cost functions for switched-parameter systems has not yet been made, there are some indications that the full generality of (24) and (25) should be restricted depending upon the various applications. If the control interval is of fixed length for a particular problem, the weighting factor, R, may as well be zero since the RT term in (24) will then only add a constant to $J_T$. In a particular subinterval (the $i^{th}$, say) the matrix $\underset{\sim}{M}_i$ would have non-zero elements for each state whose deviation from a specified trajectory

was to be penalized (i.e. increase the Cost function value). Likewise, $\underset{\sim}{N}_i$ would have non-zero elements for those elements of the control vector, $\underset{\sim}{u}(t)$, whose scaled magnitude was to contribute to the cost in the subinterval. A similar statement applies to $\underset{\sim}{H}_i$ and the deviation of the values of the state variables at the endpoint of the subinterval from some specified goal, $\underset{\sim}{x}_D(t_{i+1})$. Of course, if the value or variation of a system variable is to be ignored, the corresponding element in the appropriate weighting factor matrix is set to zero.

The Cost function measure of system quality may be used in a function minimization scheme to optimize the variable parameters in the system. In so doing, one treats $J_T$ as a function of the parameters to be varied and minimizes the function to obtain "optimum" values of the parameters. For example, one might optimize the switching instants (the $t_i$, for $i = 1,N$) for a particular switched-parameter system with a particular input, or one might try to minimize the total control interval necessary for the system to complete its response to another input (by suitably varying the $t_i$, or the initial condition due to an isolated state, perhaps), or attempt to optimize one of the system parameters.

In general, the possible values of the independent variables of the minimization will be limited (e.g. the $t_i$ may be constrained such that $0 \leq t_i \leq T$, for $i = 1,..,N$) and the minimization of $J_T$ must use those procedures applicable to constrained systems. It may be noted, in passing, that

the control vector for the switched-parameter system may be chosen so as to minimize a functional of the form of $J_T$. This represents the optimal control approach to determining the control function for switched-parameter systems. It is an area that remains to be explored.

As examples of their use, two Cost functions were constructed for the system of (19) and (20) and Figure 2. The first Cost function was

$$J_T = \begin{bmatrix} x_1(T_c) - 0 \\ x_2(T_c) - 10 \\ x_3(T_c) - 0 \end{bmatrix}^T \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1(T_c - 0 \\ x_2(T_c) - 10 \\ x_3(T_c) - 0 \end{bmatrix} . \quad (26)$$

$J_T$ was treated as a function of the length of the first subinterval, $t_1$. Successive solution of the state equations, (19) and (20), over the control interval $T_c = 10$, and evaluation of (26) for 100 incremental values of $t_1$ spanning the interval $[0,2]$, with a 10 volt step function input to the system produced the points for the curve plotted in Figure 8.

Figures 9 and 10 show the Cost function of (26) modified so that deviation from the desired final value during the entire second subinterval is counted as part of the cost. The modified Cost function is, then,

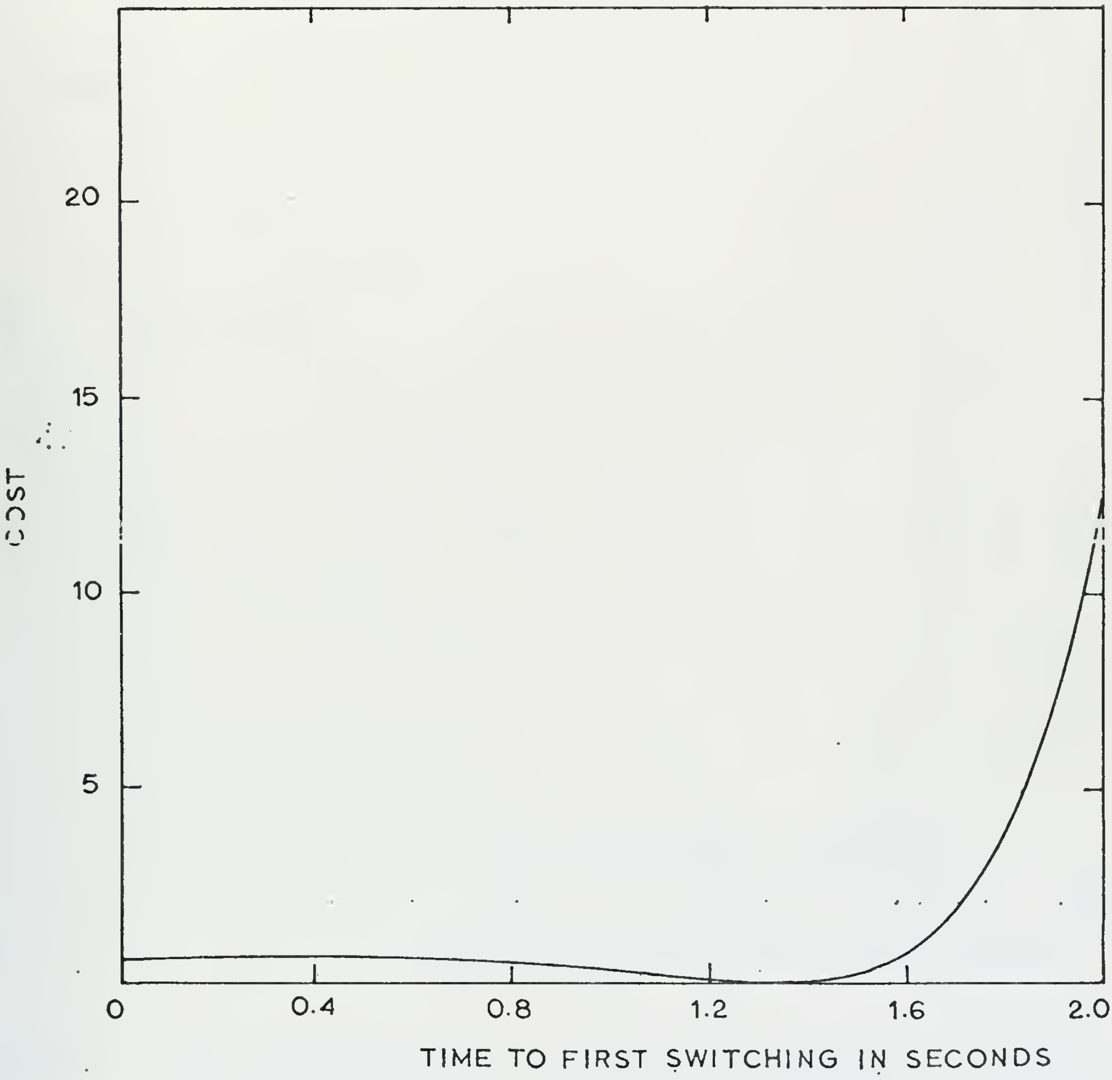


Figure 8. Cost Function of a Third Order, Switched-Parameter System as a Function of the Time to the First Switching for a 10 Volt Step Input and Where Only the Square of the Deviation of the States from the Desired Final Value is Counted as Cost.

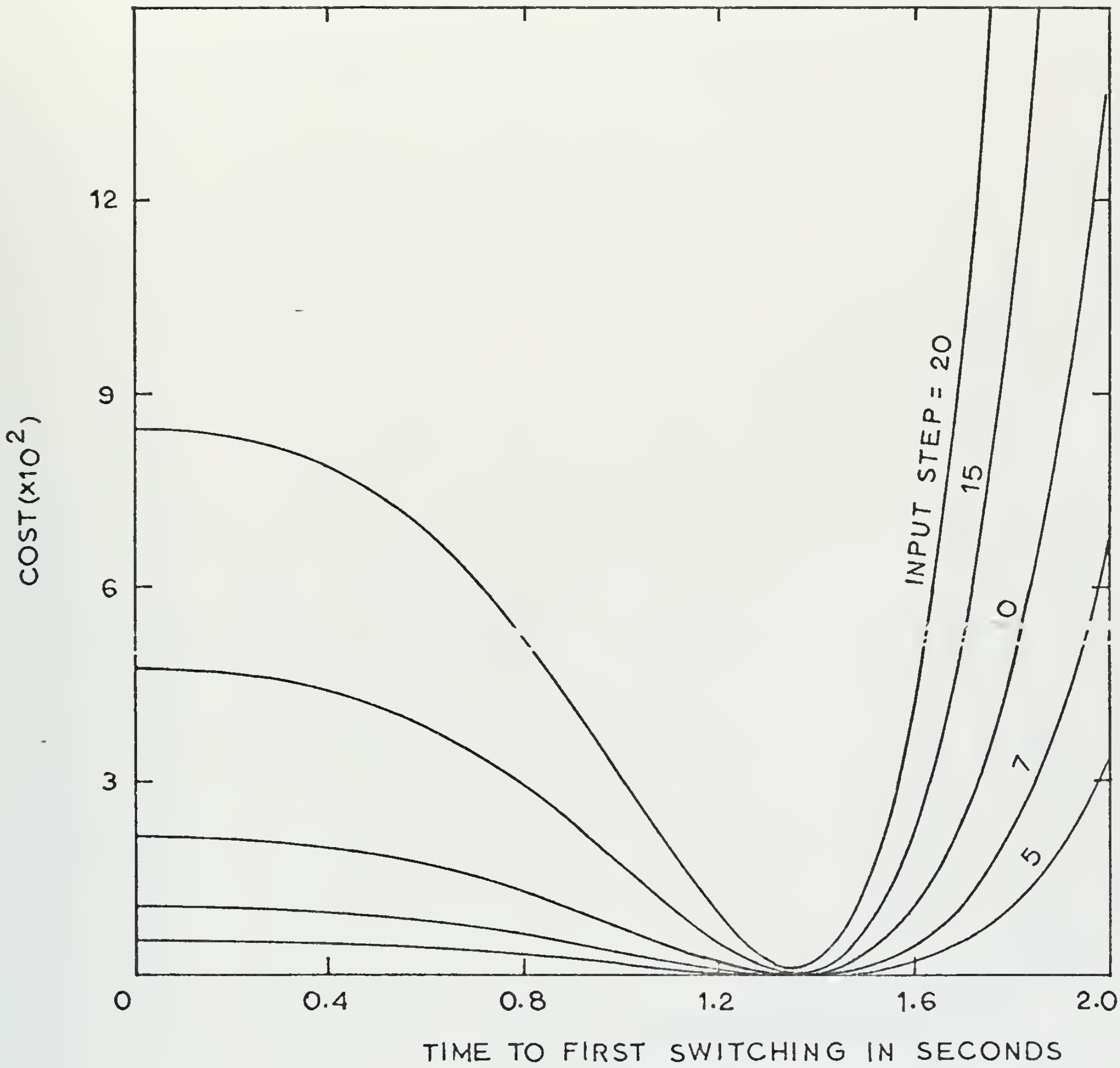


Figure 9. The Cost Function of Figure 8 Modified to Count the Time Integral of the Square of the Deviation from the Desired Final State During the Entire Second Control Subinterval as Cost, for System Input Steps of Various Amplitudes.

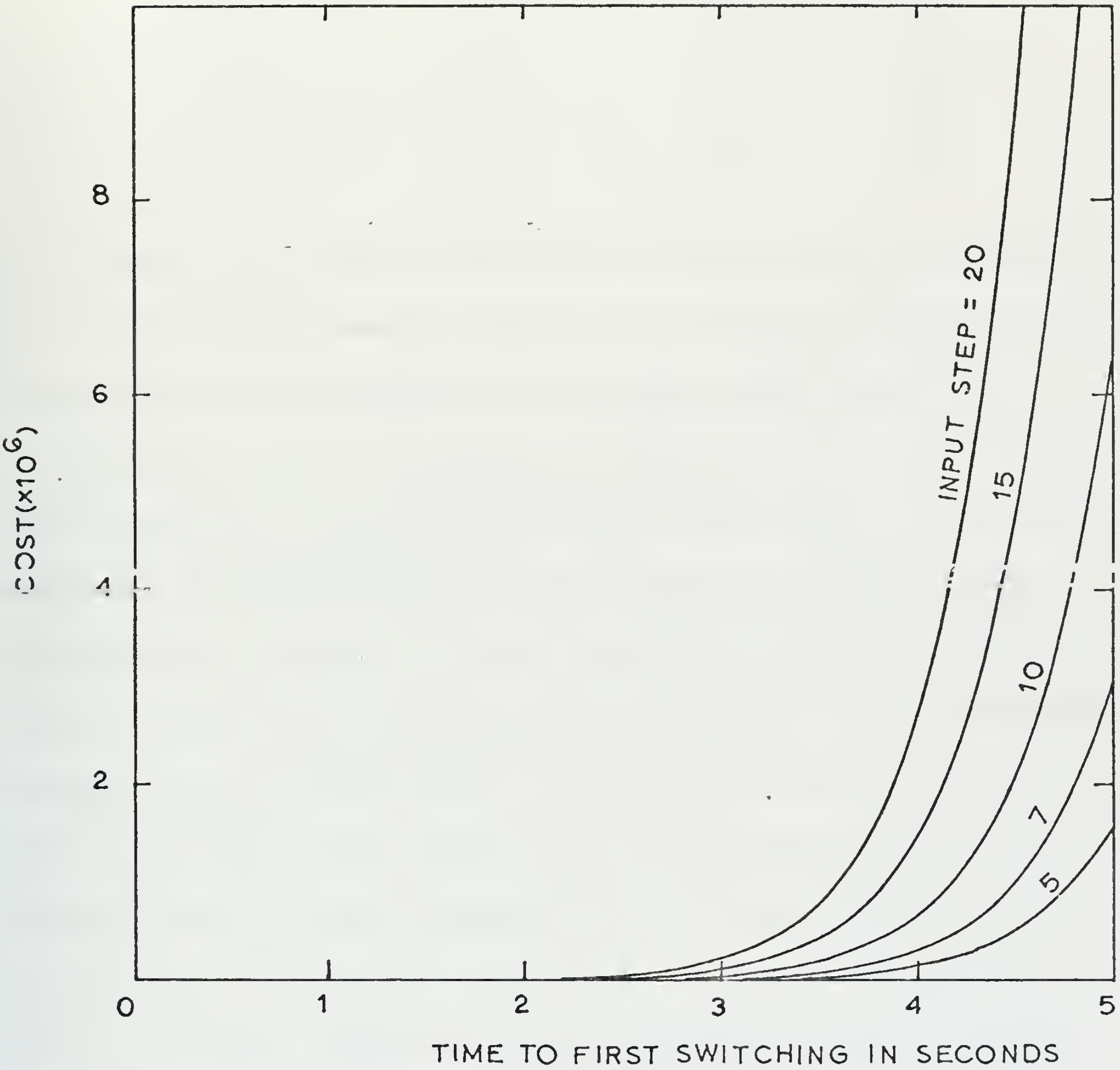


Figure 10. Same as Figure 9 Except that Time to the First Switching is Allowed to Vary in the Interval [0,5].

$$J_T = \begin{bmatrix} x_1(T_c) - 0 \\ x_2(T_c) - x_d \\ x_3(T_c) - 0 \end{bmatrix}^T \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1(T_c) - 0 \\ x_2(T_c) - x_d \\ x_3(T_c) - 0 \end{bmatrix} + \int_{t_1}^{T_c} \left\{ \begin{bmatrix} x_1(s) - 0 \\ x_2(s) - x_d \\ x_3(s) - 0 \end{bmatrix}^T \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1(s) - 0 \\ x_2(s) - x_d \\ x_3(s) - 0 \end{bmatrix} \right\} ds \qquad (27)$$

The scalar $x_d$ is both the desired trajectory (a constant) and the desired endpoint value for the second subinterval. It is also the amplitude of the input step function.

The purpose of (26) and the first term of (27) is the same; namely, to penalize (increase the value of the Cost function) the deviation of the states from their target values at the endpoint of the control interval. The Cost function, $J_T$, is increased by the square of such deviation as does exist at that time. In both (26) and (27) the values of $x_2(T_c)$ and $x_3(T_c)$ are to be penalized if they differ from $x_d$ or zero respectively. The endpoint value of $x_1(t)$ does not affect the value of the Cost function since the first element on the diagonal of the weighting factor matrix, $\underset{\sim}{H}_1$, is zero.

The second term of (27) serves to penalize the system if the trajectories of its states vary from some prescribed trajectory during (in this case) the second subinterval. Since the only nonzero element of $\underset{\sim}{M}_1$ corresponds to the deviation of $x_2(t)$ from $x_d$, only the integral of the square

of that deviation will add to the Cost function. The result is that both overshoot and slow rise time of $x_2(t)$ in the second subinterval for the switched-parameter system being considered will be penalized.

If (27) were to be applied to a shaft position control system with constant parameters, and if $x_2(t)$ and $x_3(t)$ were the state variables corresponding to the shaft position angle and its time derivative respectively, then the application of the Cost function may be seen to give a measure of how closely the system approaches the value of the input step function after some period of time which is considered as the control interval. The Cost function would also give a measure of how rapidly, without producing excessive overshoot of the final value, the system gets to that final value. It may be seen from consideration of this example that the Cost function is ambiguous in that an overdamped system which rises slowly to its final value may have the same value of cost as does an underdamped system which overshoots and then oscillates about the final value several times.

The multiple curves in Figures 9 and 10 show how the Cost function varies as the amplitude of the input step in varied. Figure 9 is analogous to Figure 8 in that $t_1$ is only allowed to vary in the interval [0,2]. Figure 10 shows how rapidly the Cost function increases as $t_1$ is increased beyond 2 and provides a non-rigorous basis for believing that the minimums evidenced by the functions in

Figure 9 are truly global minimums. The system's states were all started from zero initial conditions in the computation of these curves. The possibility of minimizing the Cost functions of Figures 8 and 9 with respect to the "variable" $t_1$ is apparent.

The determination of Cost function plots such as Figures 8, 9, and 10 requires a large expenditure of computer time since repeated solutions of the state equations are necessary to evaluate the Cost function for different values of its independent variables. Each individual curve in the three figures required approximately three minutes of computation time on the IBM 360/67 at the Naval Postgraduate School's W.R. Church Computer Center. These plots are for a single independent variable. Even more computation time would have been required for more than one independent variable. Fortunately, function minimization usually requires fewer evaluations of the Cost function and, hence, less computation time.

# V. SUMMARY AND CONCLUSIONS

The preceding material has 1) developed a general description of linear, switched-parameter systems in terms of the mathematics of state variables, 2) presented the concepts of state variable grouping and of the "core" states of a switched-parameter system, 3) presented two procedures for determining the sequence of coefficient matrices in the state variable representation that describes the behavior of the switched parameters in the system and illustrated their use with simple examples, and 4) presented one possible Cost function which may be used to measure the performance of these systems and applied it in two examples to demonstrate its application as a measure of system quality. That such a Cost function may provide a useful way to optimize the behavior of the system has been demonstrated in Appendix A.

Several conclusions may be drawn from the work just summarized and the examples which were presented in conjunction with that work. Although the model that was developed was not applied to a real-world system, it is obvious from both the mathematics and the examples that physical systems like those presented in the Introduction may, indeed, be modeled as switched-parameter systems. A conclusion that is possible as a by-product of the examples of switched-parameter systems that were used is that it is possible to

greatly improve the step response of a heavily damped second order system by using the states in the second order system as "core" states in an unstable second or third order system in order to "pre-charge" them. The penalty that one must pay for such improved response is a marked dependence of the response on the time that is taken to "pre-charge" the states.

One may also conclude that it is possible to use Cost function minimization to determine the optimum switching time which produces the "best" step function response in at least one variety of switched-parameter system. The optimum switching time so determined is dependent upon the form of the Cost function which was minimized. Finally, it is evident that the digital computer methods used to solve the state equations and to determine the optimum switching point were both sensitive to and able to cope with the sensitivity of the system response to the switching time.

This first attempt to study linear, switched-parameter systems in some generality has raised many more questions than it has answered; indeed, much the greater part of a complete study of these systems remains to be done. Not all of the possible paths that lead from this tentative beginning have been discovered. Some of those which have been and which may prove worthy of further study are presented in the following section.

# VI. RECOMMENDATIONS FOR FURTHER WORK

The areas of further study that are recommended below constitute a study of what may be called the "design problem." Some of the questions require that one attempt to answer the question, "What is the optimum system to use for this job?" That question has been answered heuristically in the past and a definitive answer is not expected soon. Still, one may ask questions about the system architecture that do have some hope of being answered.

The first problem relating to the architecture of the systems that might be studied would be to define the constraints on the physical system that are required to obtain a certain order of behavior of the "core" states. For instance, one might ask what system architecture is required to produce third order behavior in a system with two "core" states.

The kind of behavior that one might seek to establish in a switched-parameter system in order to satisfy the design goals should be related to what is possible, desirable and obtainable in the way of system architecture. This might be approached by seeking to determine what sort of output responses are possible when one switches between two particular linear systems. If one were to form a table of systems before and after switching, and the particular response characteristics obtained, one might enter the system

used in Figure 3, noting that it goes from a third order system with a pole in the right half of the s-plane to a damped second order system and that the result is an improvement in the system step response for a suitable selection of switching time. The hope is that [by looking at such a table], one would be able to decide which system to switch to in order to obtain a desired response.

A "building block" approach may also be possible. From the table just mentioned it may be possible to identify certain sequences of linear systems which have characteristics such that one might desire to use these subsequences as building blocks in constructing various systems. Of course, this raises the question of determining what sorts of building blocks might be required, or desired, for such a system and how to put them together to obtain the desired results.

The work of Flügge-Lotz and Taylor [6] that was mentioned in the introduction indicates that a quasi-adaptive scheme can yield useful results. A generalization of those results in terms of switched-parameter systems would be desirable.

For switched-parameter systems generally, one might inquire what the switching criteria should be. As an example one may not want to switch to a particular system while a capacitor has a non-zero charge. One might switch instead (with the switching based on the values of some other states) to an intermediate system which would discharge the capacitor before making the transition to the particular system first mentioned.

As a grand goal of the work with system architecture, one might seek a procedure for the systematic design of a switched-parameter system to meet some specified requirements.

Optimization problems represent another area of study that could produce significant results. One usually encounters such problems in connection with optimizing an input to an already fixed system; however, with a system which is not fixed (e.g. switched-parameter) it may be reasonable to ask what sequence of systems gives an optimum response for a fixed input. This has the nature of an inside-out optimal control problem where one seeks to determine the system that makes a given control optimal.

The possibility of varying the initial conditions of certain of the states in the system was shown in Section III to allow a large improvement in the response of a heavily damped second order system, and has been mentioned as a possible candidate for an optimization technique. This idea should be explored further, as should the theoretical improvement in system response that may be attained. Being able to vary the initial condition of one of a system's states would seem to permit one to use the transient response in the system caused by that initial condition to alter the transient response of the other states of the system in a controlled fashion.

The kind of Cost function that one uses will influence the results of the optimization of a system. The

correspondence between changes in the Cost function and changes in the physical response of the system is very important. As was seen in the various examples, a smooth, slowly varying Cost function may correspond to a physical response which is extremely sensitive to the particular parameter variation, as was the case with $t_1$. The establishment of guidelines for the assignment of weighting factor values, thus, seems useful. It may be worthwhile to investigate the type of Cost function necessary to allow a system parameter to be optimized with the result that some of the system's states may be constrained to stay within certain limits. This would be useful since it would allow the designer to specify physical limits on the system's states where such limits resulted from saturation limits on the physical device or from some desired performance criteria.

It should be possible to formulate an optimal control problem for switched parameter systems. This should be determined and any limitations on the system imposed by the requirements for the existance and uniqueness of the optimal control systems with constant parameters and with the "optimal" control system mentioned previously in which the control is obtained by switching parameters.

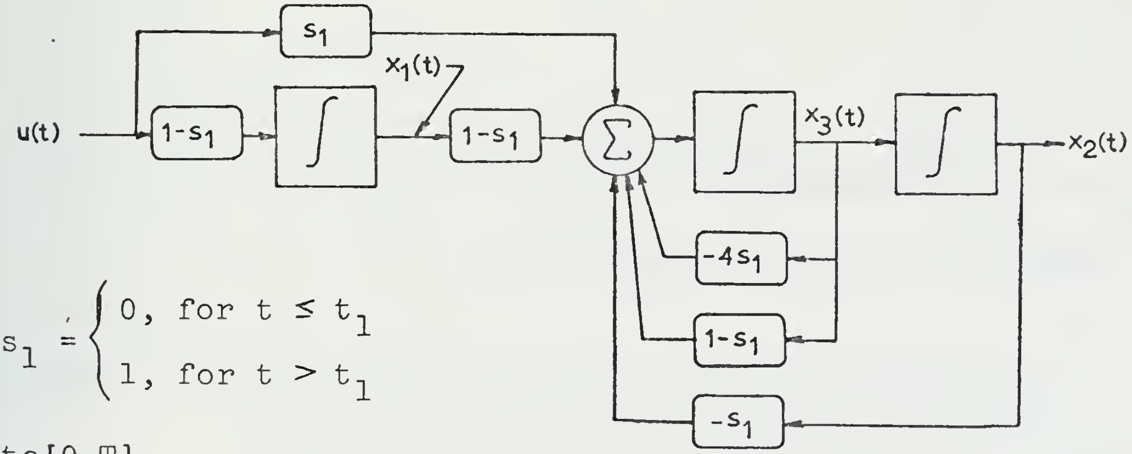
In describing the switching curves of the present relay control systems, it is common practice to speak of the switching boundary as a hyper-surface in state space. Further work might be done to determine the effects (beneficial or deleterious) of being able to switch on a particular

"line" in that hyper-surface, where such flexibility would be allowed by the independent control of the initial condition of a particular state which was isolated from the "core" states at some point. Such considerations could also be the basis for the design of certain small segments of a larger system.

# APPENDIX A

## APPLICATION OF COST FUNCTION MINIMIZATION

The purpose of this appendix is to demonstrate the utility of a Cost function minimization procedure as a means of optimizing the step function responses of some simple switched-parameter systems. As the first example of a Cost function to be minimized, consider Figure 12. The set of functions shown there resulted from the application of five different amplitude step functions to the switched-parameter system of Figure 11, which is the system of Figure 2 reproduced here for convenience. The system was started from zero initial conditions for all states in each case.



$$s_1 = \begin{cases} 0, & \text{for } t \le t_1 \\ 1, & \text{for } t > t_1 \end{cases}$$

$t \varepsilon [0,T]$

Figure 11. Replica of Figure 2. Block Diagram of a Third Order, Switched-Parameter System.

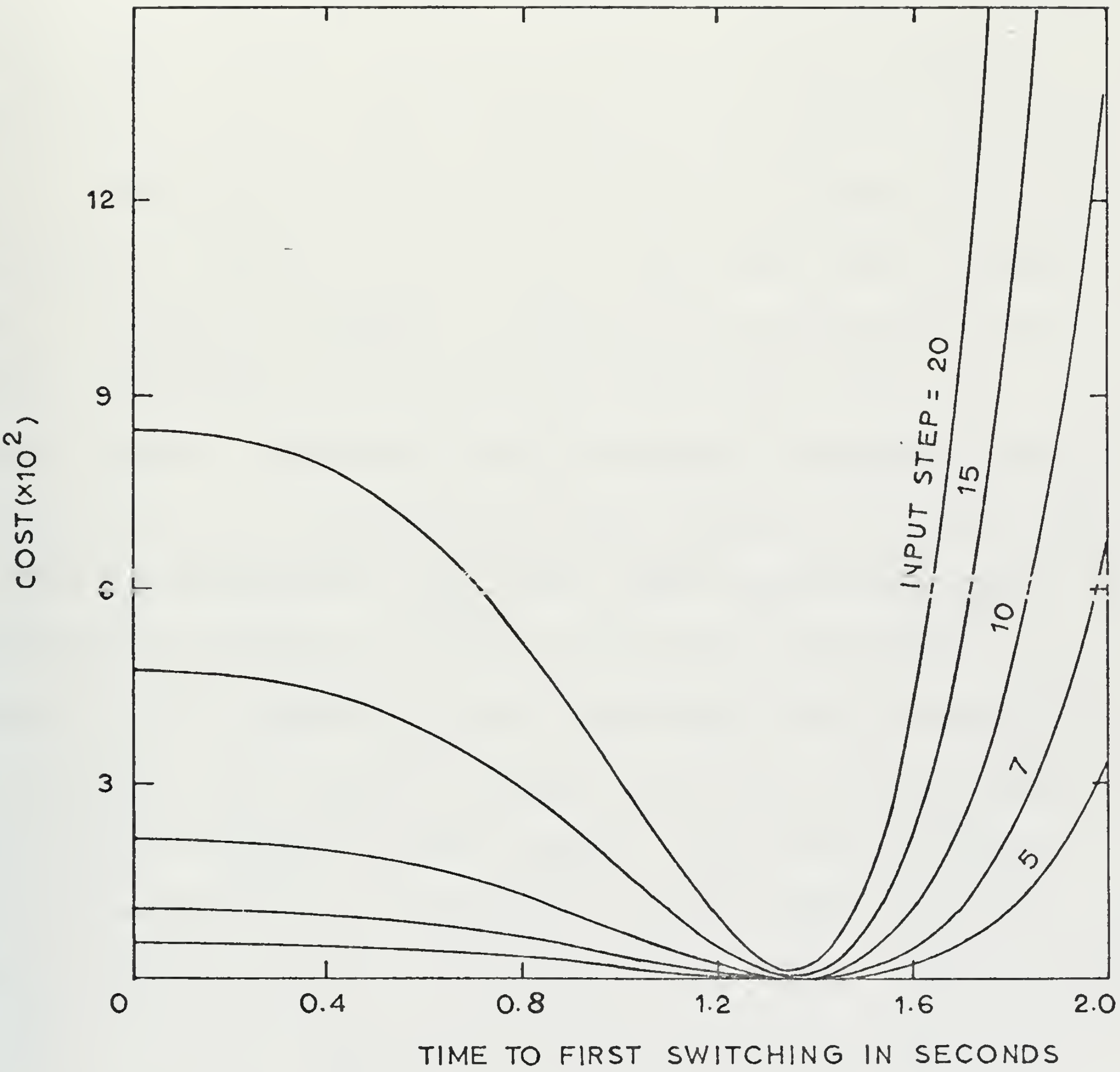


Figure 12. Replica of Figure 9. The Cost Function of a Third Order, Switched-Parameter System as a Function of the Time to the First Switching for Various Amplitude Step Inputs; where the Square of Both the Deviation of the States from the Desired Endpoint Values and the Deviation of the States from the Desired Final State During the Second Subinterval are Counted as Cost.

From Figure 12 it is easily seen that the Cost functions each have a minimum for a time to first switching, $t_1$, of about 1.35 seconds. For low amplitude steps (e.g. 5 volt, or 7 volt) the minimum is quite broad. It becomes more sharp as the step amplitude increases to 20 volts. Because of the broad minimum, one would expect the time response of the system not to vary drastically for a small change in $t_1$. A comparison of Figures (13(a) and (b) (which are replicas of Figures 3(a) and (b)) shows that this is the case. The small increase in the Cost function which is barely discernable on the 10 volt input step curve of Figure 12 as $t_1$ increases to 1.45 seconds, is seen to be reflected in an overshoot-and-decay response in Figure 13(a).

Not to be ignored is the fact that the various Cost functions of Figure 12 all have about the same minimum. Hence, for this system, if the switching time is taken as 1.35 seconds, the system response should be near optimum for a wide variation of step inputs. The system behaves as a linear system, the character of its output response being unaffected by the amplitude of the input. This linear response is to be expected since the system is linear in both subintervals and the switching instant, $t_1$, is independent of the values of the states, being fixed in value. This is brought out vividly in Figures 14, 15, and 16, which show the output response of this system for input steps of 5, 7, and 15 volts respectively.

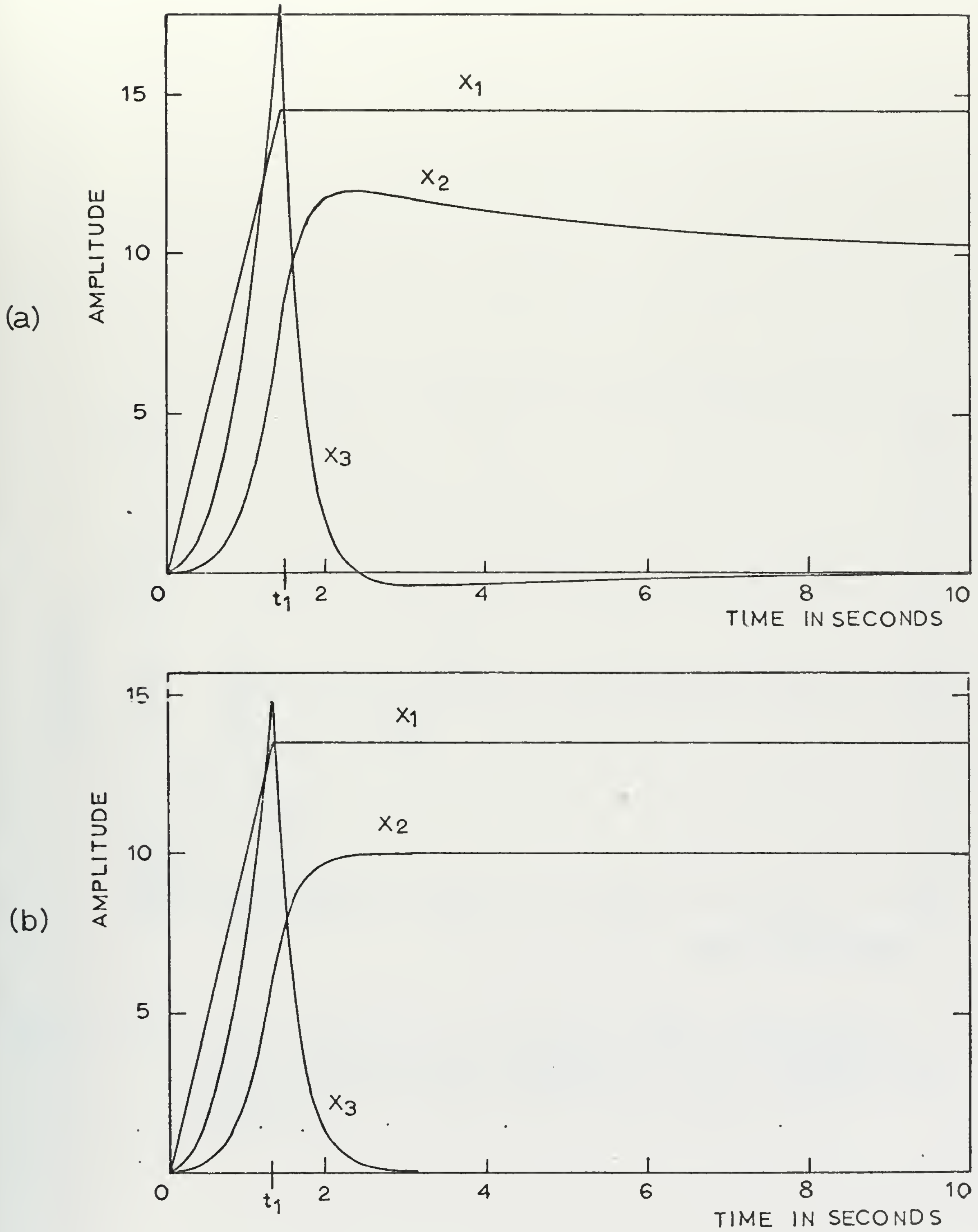


Figure 13. Replica of Figure 3. Time Response of a Third Order, Switched-Parameter System to a 10 Volt Step Input. (a) Switching Time, $t_1$ = 1.45 Seconds. (b) $t_1$ = 1.349 seconds.

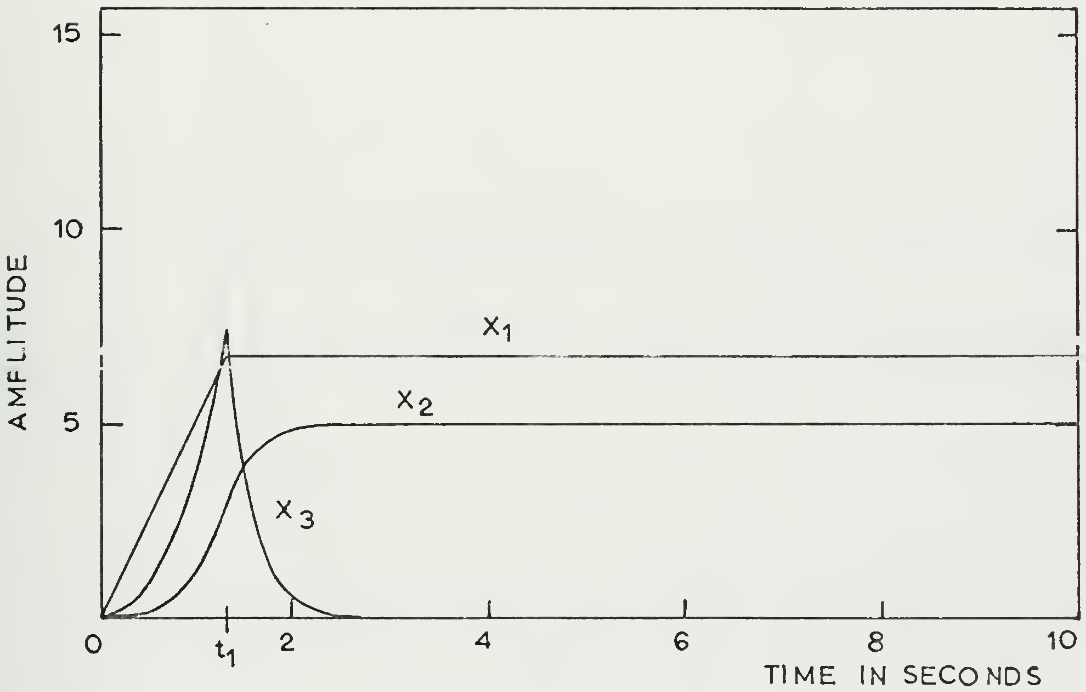


Figure 14. Time Response of the Third Order, Switched-Parameter System of Figure 11 to a 5 Volt Input Step with the Switching Time, $t_1$ = 1.3475 Seconds.

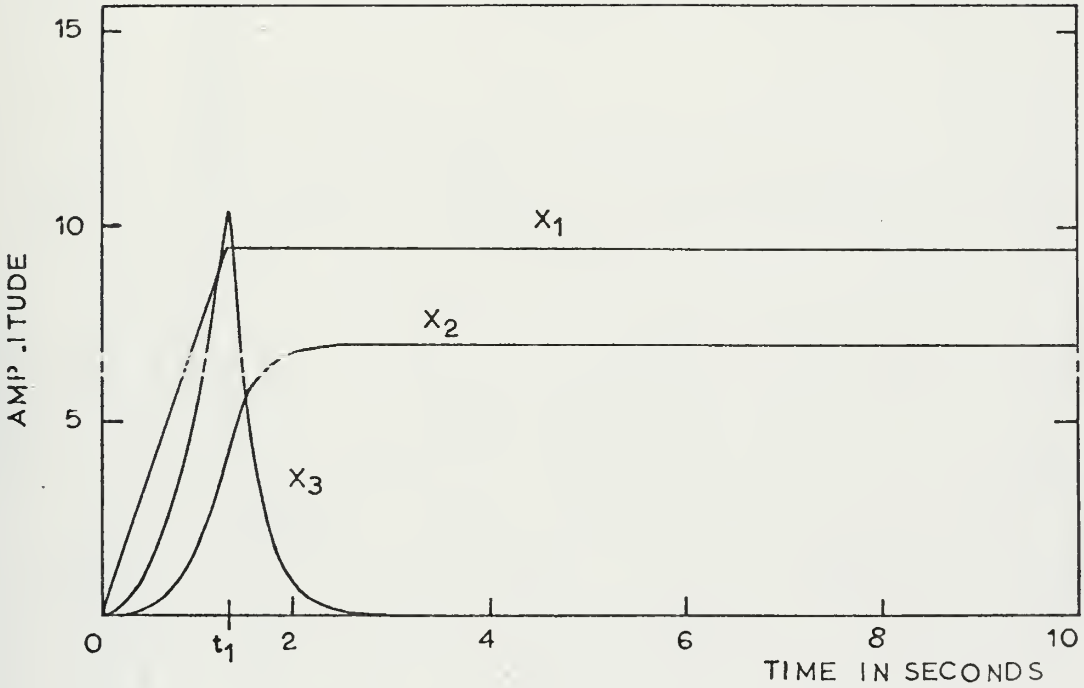


Figure 15. Time Response of the Third Order, Switched-Parameter System of Figure 11 to a 7 Volt Input Step with the Switching Time, $t_1$ = 1.3475 Seconds.

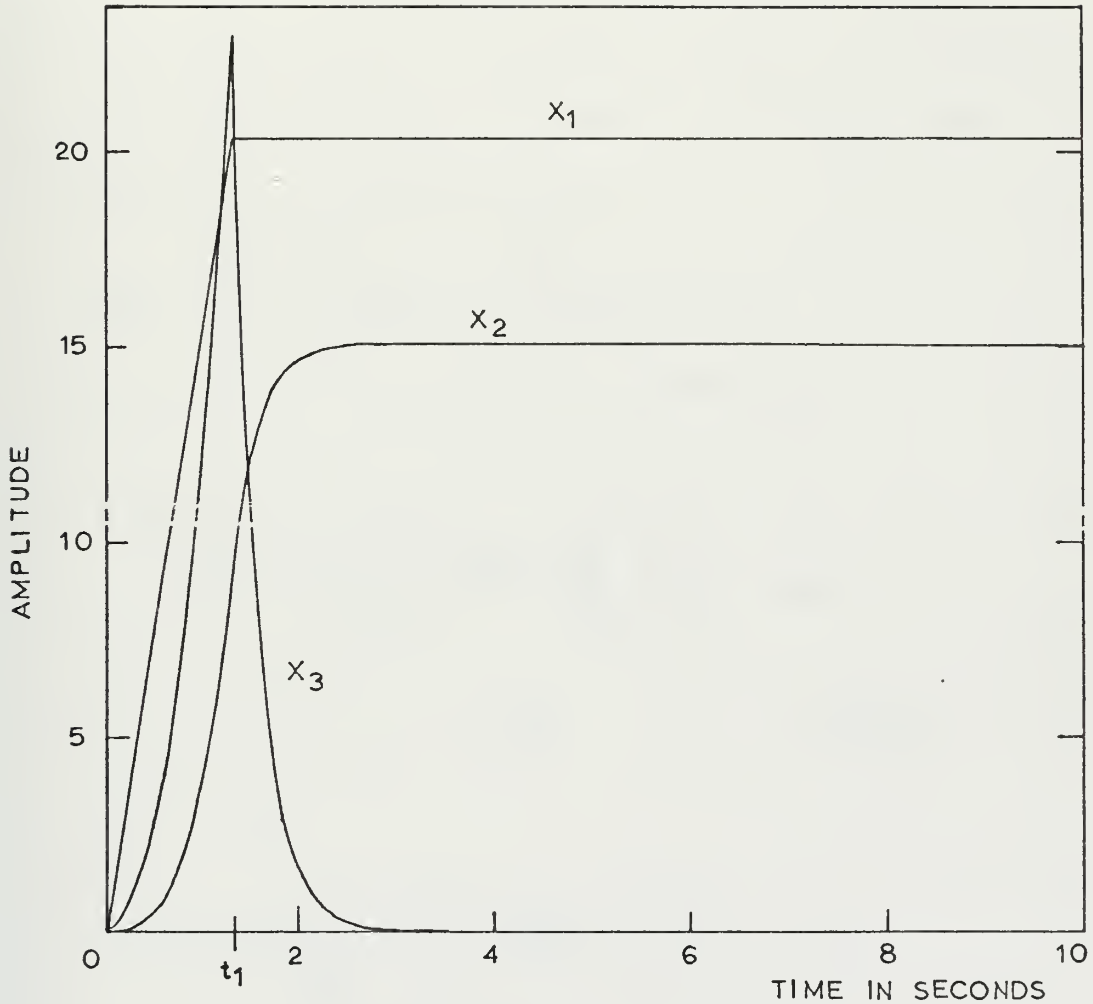


Figure 16. Time Response of the Third Order, Switched-Parameter System of Figure 11 to a 15 Volt Input Step with the Switching Time, $t_1$ = 1.3475 Seconds.

These figures should be compared with Figures 13(a) and (b). Note that there is a slight change in the value of $t_1$ from 1.349 in Figure 13(b) to 1.3475 for Figures 14, 15, and 16.

The second example of the application of Cost function minimization will be for a second order system with a switched damping factor. The block diagram of the system before and after switching is shown in Figure 17. The state equations for this system are

$$\dot{\underset{\sim}{x}}(t) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \underset{\sim}{x}(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} [u(t)] \text{ for } t \varepsilon [0, t_1],$$

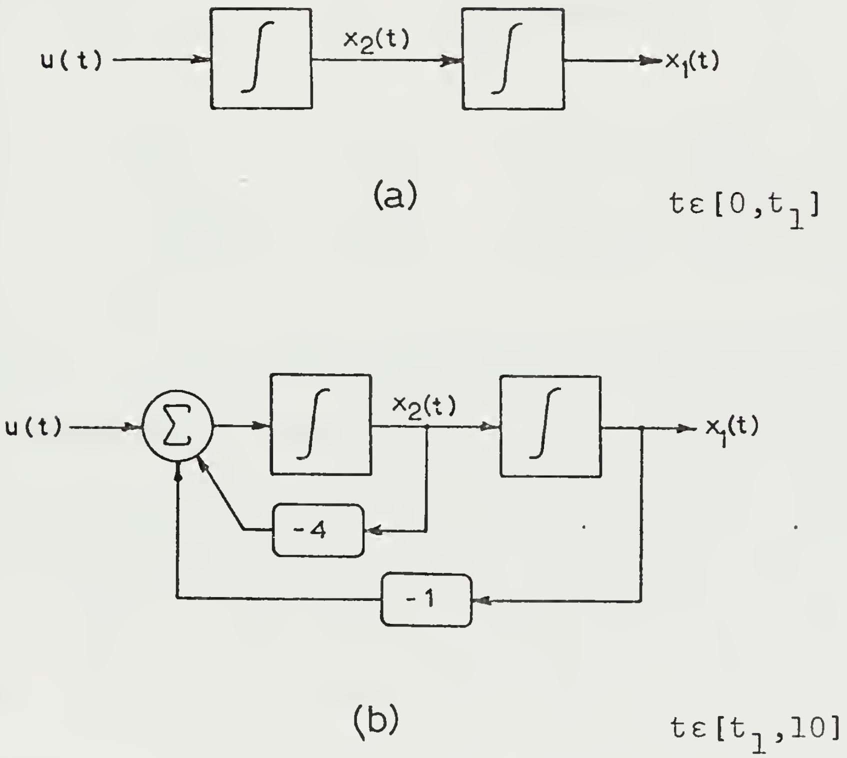


Figure 17. Block Diagram of a Second Order, Switched-Parameter System. (a) Before Switching. (b) After Switching.

and

$$\underset{\sim}{\dot{x}}(t) = \begin{bmatrix} 0 & 1 \\ -1 & -4 \end{bmatrix} \underset{\sim}{x}(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} [u(t)] \text{ for } t \varepsilon [t_1, 10],$$

where the length of the control interval is taken as 10 seconds.

A Cost function for this system was constructed analgous to the one for the third order system just discussed, the function being dependent on the final values of the states at the end of the control interval and dependent upon the deviation of the state $x_1(t)$ from the desired final value during the second subinterval. Thus, the Cost function was

$$J_T = \begin{bmatrix} x_1(10)-x_d \\ x_2(10)-0 \end{bmatrix}^T \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1(10)-x_d \\ x_2(10)-0 \end{bmatrix} + \tag{28}$$

$$\int_{t_1}^{10} \left\{ \begin{bmatrix} x_1(s)-x_d \\ x_2(s) - 0 \end{bmatrix}^T \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1(s)-x_d \\ x_2(s) - 0 \end{bmatrix} \right\} ds$$

The first term of (28) penalizes the deviation of the states from their desired values at the end of the control interval by increasing the Cost function by the sum of the squares of the deviations of both states from the desired end point values. Deviation of $x_1(t)$ from the input step amplitude (the desired final state for $x_1$) represented by $x_d$, and $x_2(t)$ from zero final amplitude are both penalized in this way. The second term penalizes any deviation of

$x_1(t)$ from the desired trajectory (in this case, a constant value, $x_d$, which is also the desired final state) during the second subinterval, by adding the integral of the square of the deviation to the cost during the subinterval. By penalizing the deviation from the desired trajectory in this way it is hoped that the optimization procedure can be forced to choose a switching time, $t_1$, which will give the system a rapid rise time and little overshoot. This Cost function is plotted as a function of the time to the first switching, $t_1$, in Figure 18. Zero initial conditions were imposed upon $x_1(t)$ and $x_2(t)$ in determining this Cost function.

In order to test the usefulness of the function minimization approach, the Cost function, $J_T$, was minimized as a function of $t_1$ using a Fortran IV subroutine which implements the Tangent Search method for the minimization of a function with constraints. See [9] and [10] for further details of the program and the method.

The result of the optimization is shown as Figure 19, which shows the "optimum" time response of this second order, switched-parameter system as the functions $\hat{x}_1$ and $\hat{x}_2$. These response curves should be compared to those of the constant-parameter second order system that are shown as $x_1$ and $x_2$ which are taken from the response curves of Figure 4. The improvement is evident, as is the price that must be paid for "snappy" response; namely, that the time response of the velocity ($\hat{x}_2(t)$) becomes impulse-like, indicating that the system of $\hat{x}_1$ and $\hat{x}_2$ is undergoing some

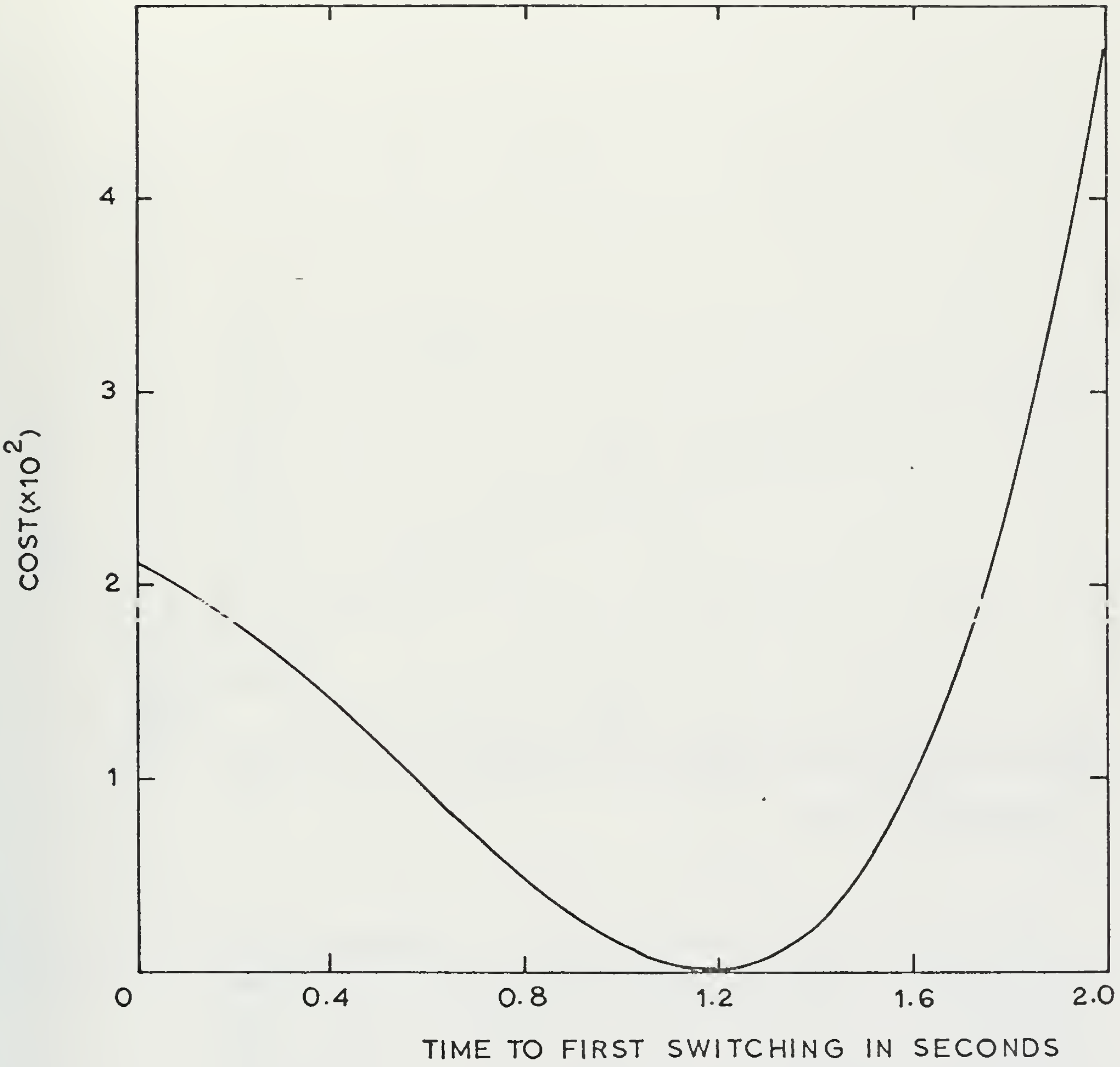


Figure 18. Cost Function of the Second Order, Switched Parameter System of Figure 17 as a Function of the Time to the First Switching for a 10 Volt Step Input.

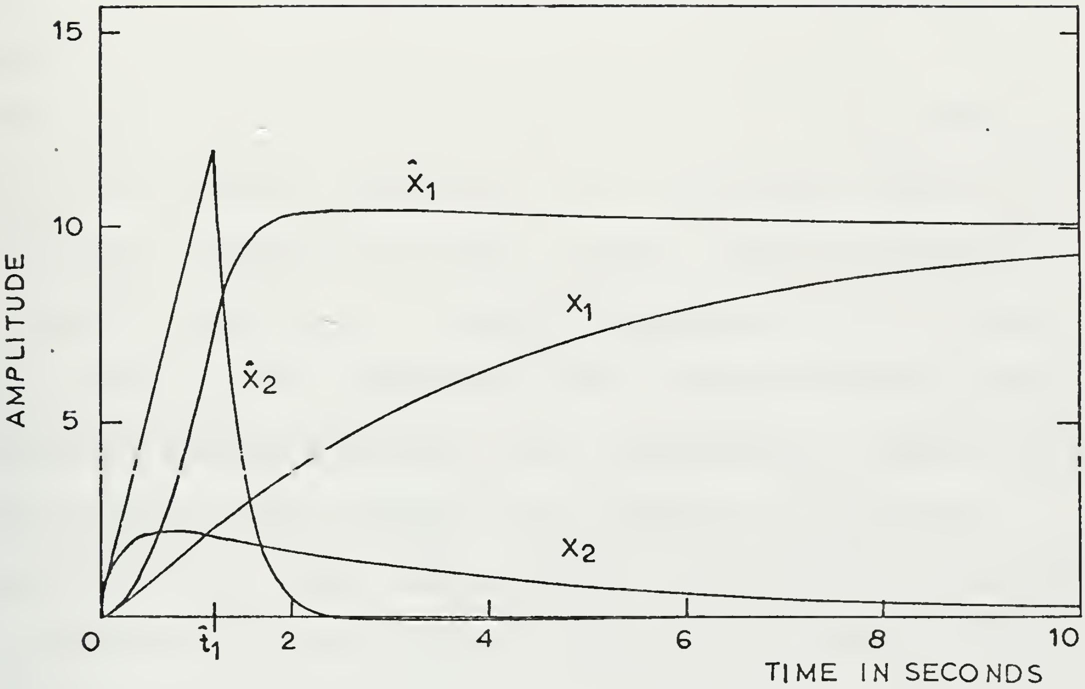


Figure 19. Time Response of the Second Order, Switched-Parameter System of Figure 17 After the Switching Time, $t_1$, has been Optimized. The Optimum Value, $\hat{t}_1 = 1.216$ seconds. $\hat{x}_1$ and $\hat{x}_2$ are the Responses of the States of the Optimized System. $x_1$ and $x_2$ are the Responses of a Related Constant Parameter System. The Input is a 10 Volt Step in Both Cases.

relatively violent accelerations compared to the soft response of the constant-parameter, linear system.

The final example demonstrates the improvement that can be made in the step response of the third order, switched-parameter system of Figures 5 and 6 (shown below as Figure 20). The time responses of this system to a 10 volt step input, for two arbitrary switching times are shown in Figure 7 which is reproduced below as Figure 21.

Two Cost functions were constructed for this system. As for the previously mentioned Cost functions, these two were plotted assuming that the system's states started at zero initial conditions. One was dependent on the values of the states at the endpoint of the control interval only, analgous to (26) and Figure 8 for the system of Figure 11; and the second Cost function was analgous to (27) and Figures 9 and 10. The Cost function which depends only on the deviations of the endpoint values of the states from some desired final value is shown in Figure 22 for a system with a 10 volt step input. It is seen to be not markedly different from Figure 12 for the 10 volt step curve except that the minimum value is displaced slightly to a larger value of $t_1$. The minimum is seen to be for $t_1$ slightly less than 1.6 for an essentially zero value of the Cost function. Although it is not evident from Figure 22, the Cost function continues its rapid rise as $t_1$ is increased beyond $t_1 = 2.0$ in the same manner as that shown in Figure 10, implying that the minimum at $t_1 = 1.6$ is a global minimum.

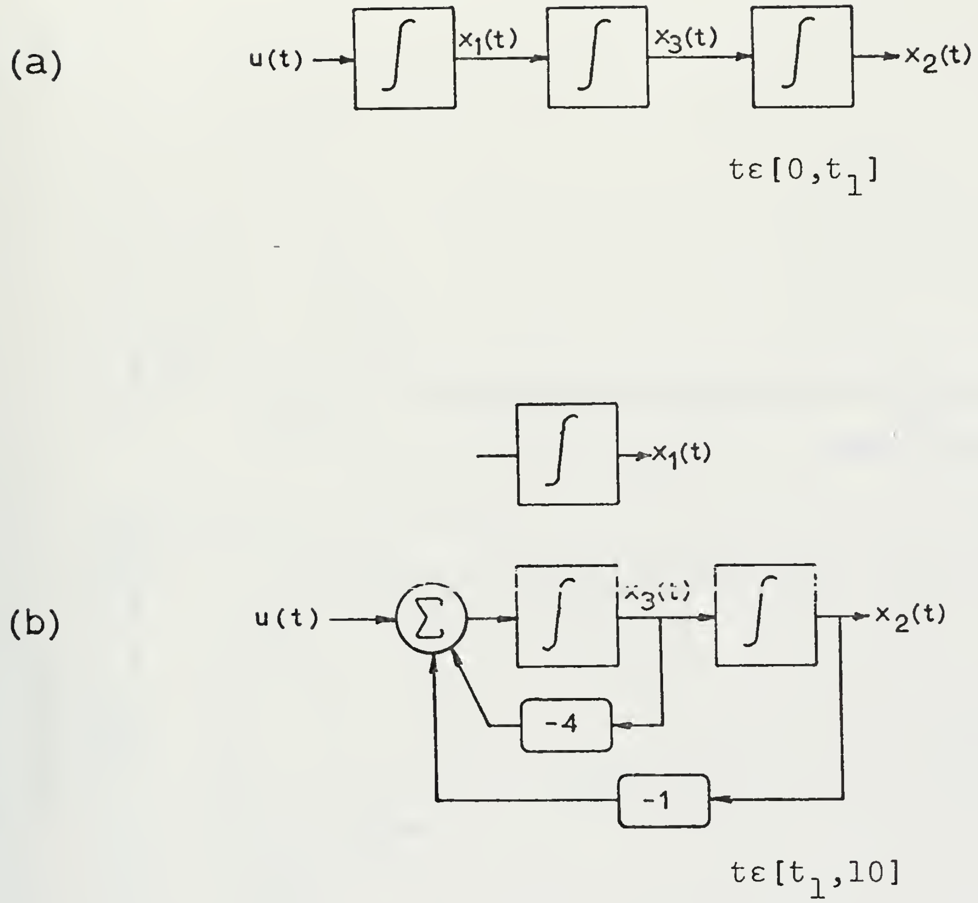


Figure 20. Replicas of Figures 5 and 6. Block Diagrams of a Third Order, Switched-Parameter System. (a) Prior to Switching. (b) After Switching.

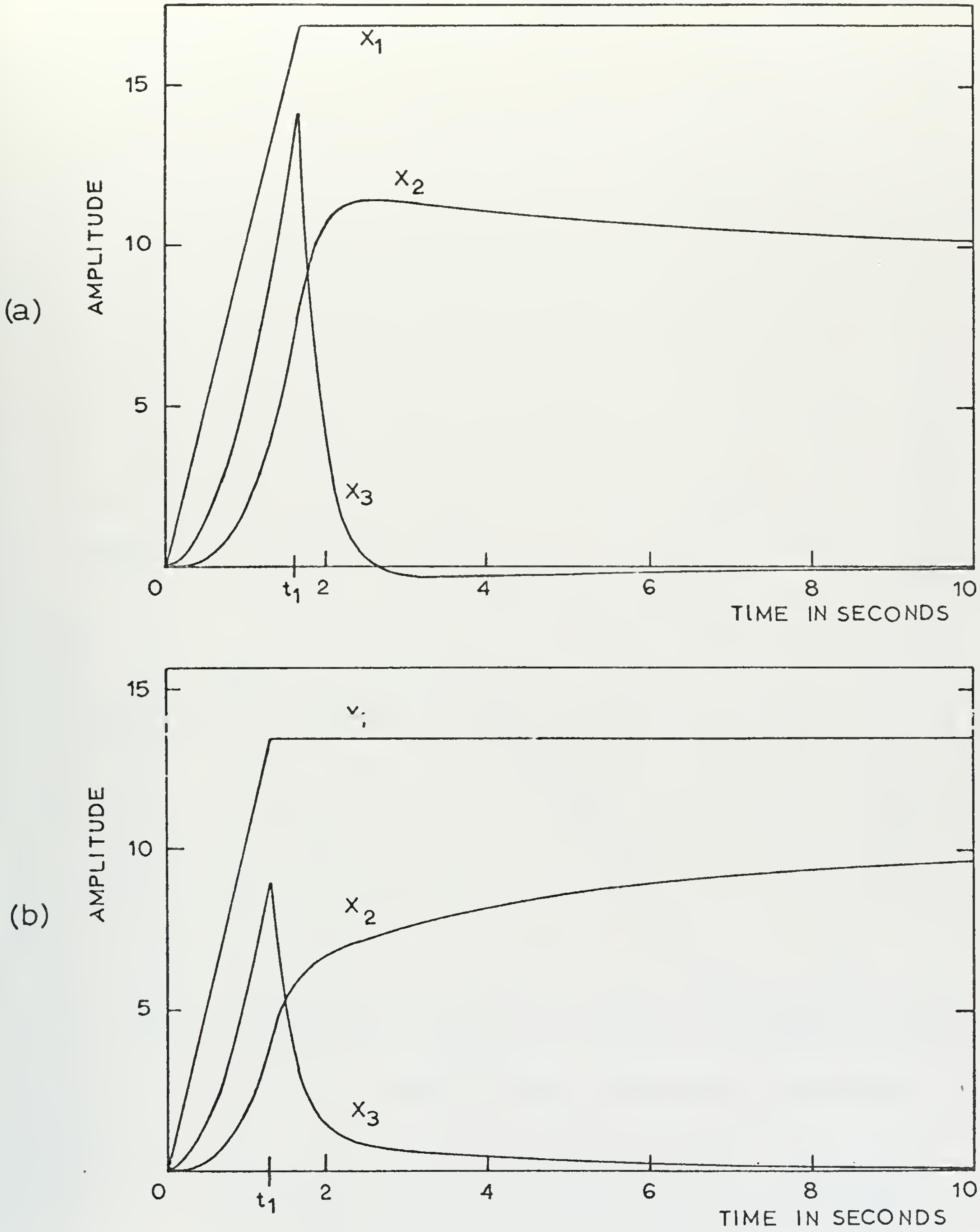


Figure 21. Replica of Figure 7. Time Response of the Third Order, Switched-Parameter System of Figure 20 to a 10 Volt Input Step. (a) Switching Time, $t_1$ = 1.699 Seconds. (b) $t_1$ = 1.349 Seconds.

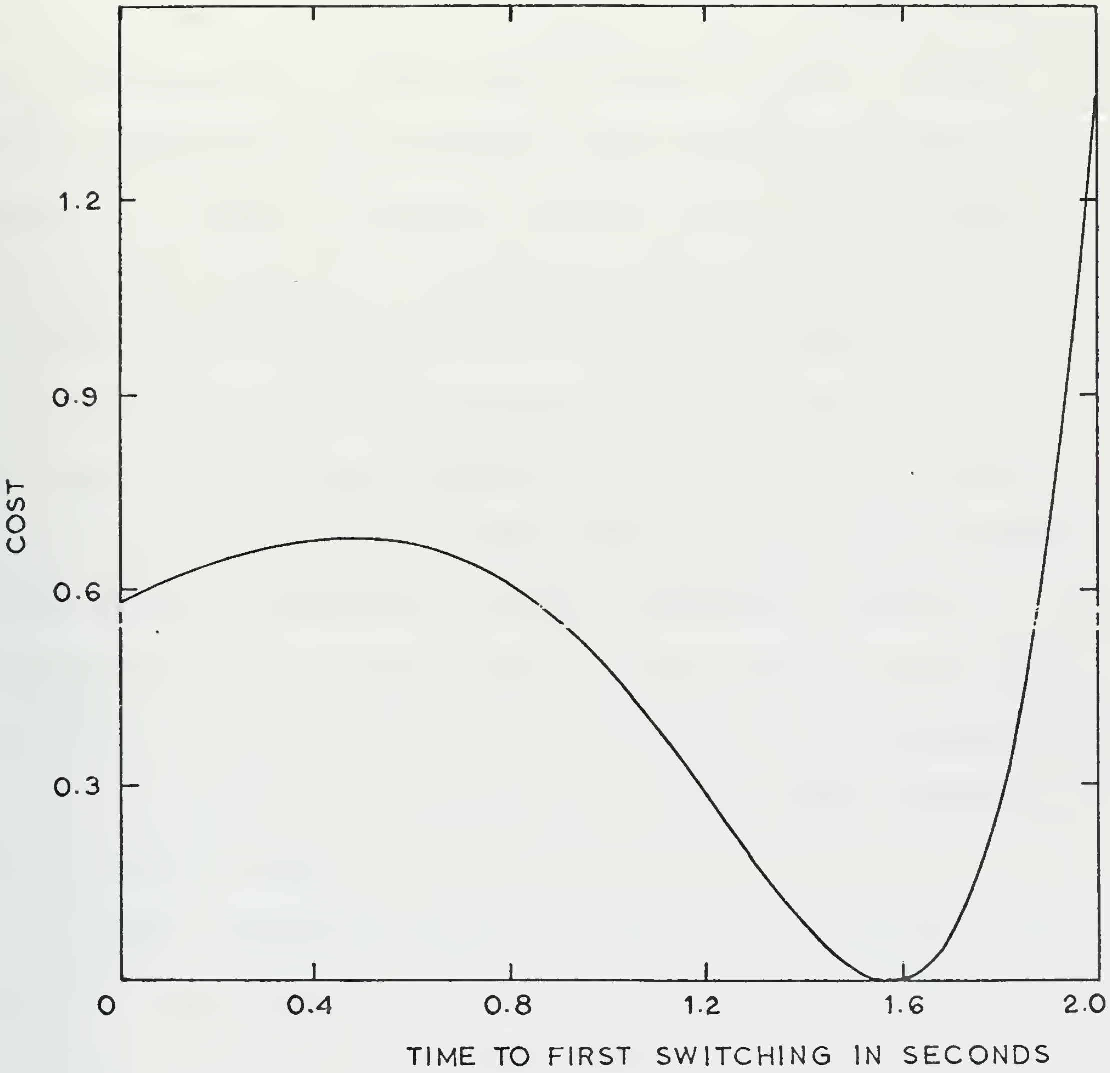


Figure 22. Cost Function for the Third Order, Switched-Parameter System of Figure 20, as a Function of the Time to the First Switching for a 10 Volt Step Input; Where only the Square of the Deviation of the States from the Desired Final Value is Counted as Cost.

This Cost function was minimized using the same method as was used for the second order system just considered with the addition that three different values of step input amplitude were used. The "optimum" switching time was found to be the same for all of the inputs, $t_1 = 1.562$ seconds. The time responses of the output states, $x_2(t)$, for the optimum response to the various step inputs are shown in Figure 23. Figure 24 shows a similar plot of the time responses of $x_3(t)$, the derivatives of the output states. It should be noted from Figure 23 that the character of the optimum response is independent of the input step amplitude, the various responses differing only in magnitude. This result is not unexpected. It is a reflection of the system's linearity in each subinterval and the independence of $t_1$ of the values of the states during the control interval, just as the third order system considered at the beginning of this appendix showed linear behavior for the same reasons.

In order to obtain some feeling for the influence the choice of Cost function has on the "optimum" response obtained, or, what is the same thing, on the value of $t_1$ produced by the minimization of the Cost function, another Cost function was constructed for the same system. This new function was the analog of (27) and Figures 9 and 10. The new Cost function is plotted as a function of the time to the first switching in Figure 25. There, it may be noted that the "optimum" value of $t_1$ is not the same as

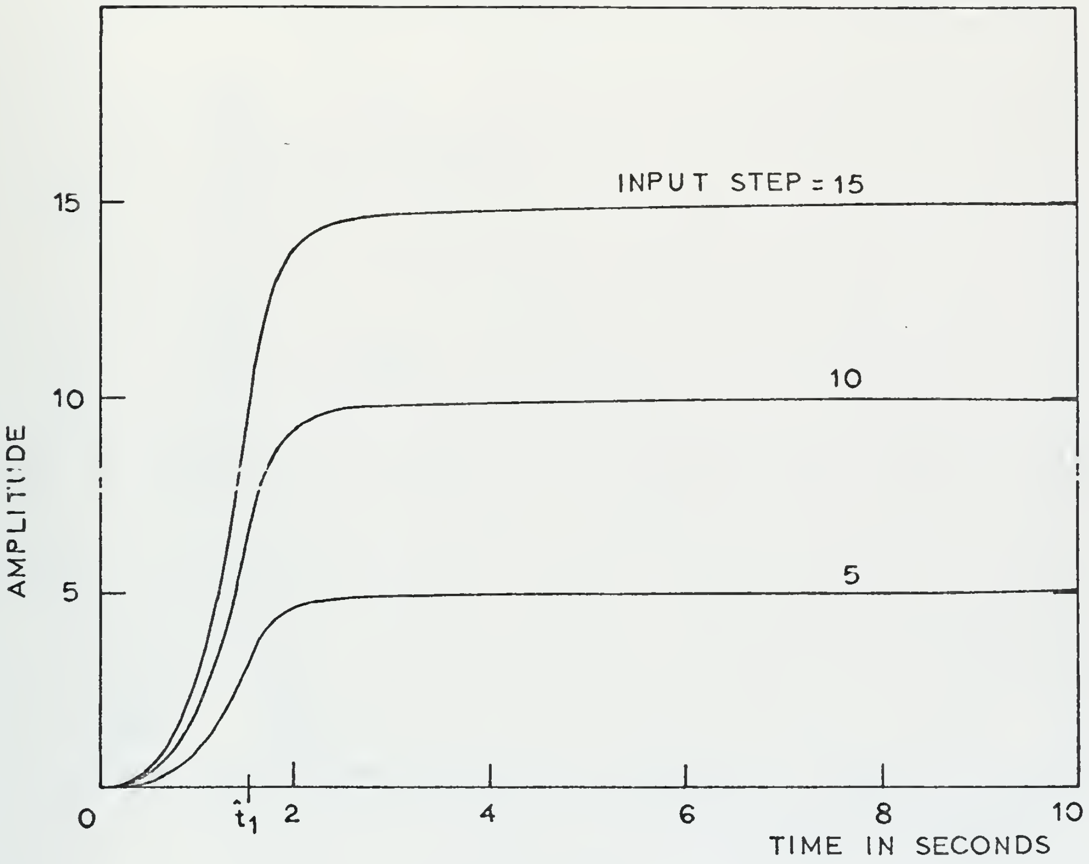


Figure 23. Optimum Time Response, $\hat{x}_2(t)$, of the Third Order, Switched-Parameter System of Figure 20 Obtained by Minimizing the Cost Function of Figure 22 with Respect to the Time to the First Switching, for Various Amplitude Input Steps. Optimum Switching Time, $\hat{t}_1$ = 1.562 Seconds.

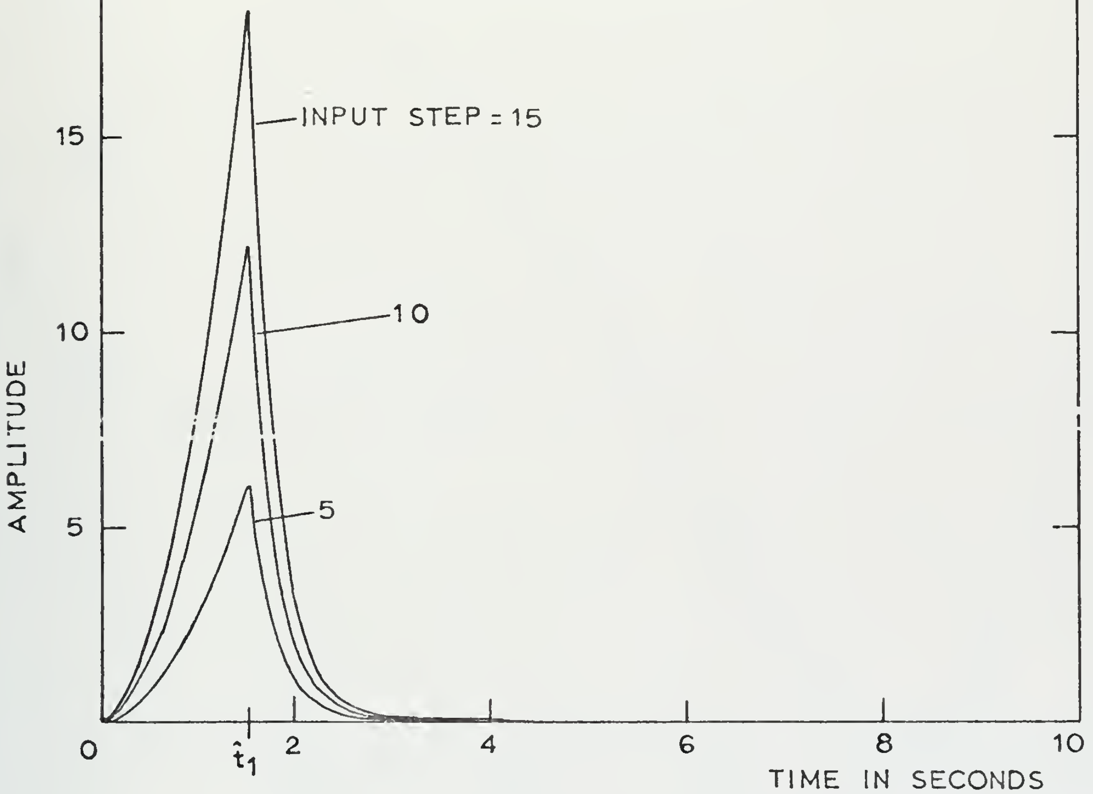


Figure 24. Optimum Time Response, $\hat{x}_3(t)$, for the Same System as in Figure 23.

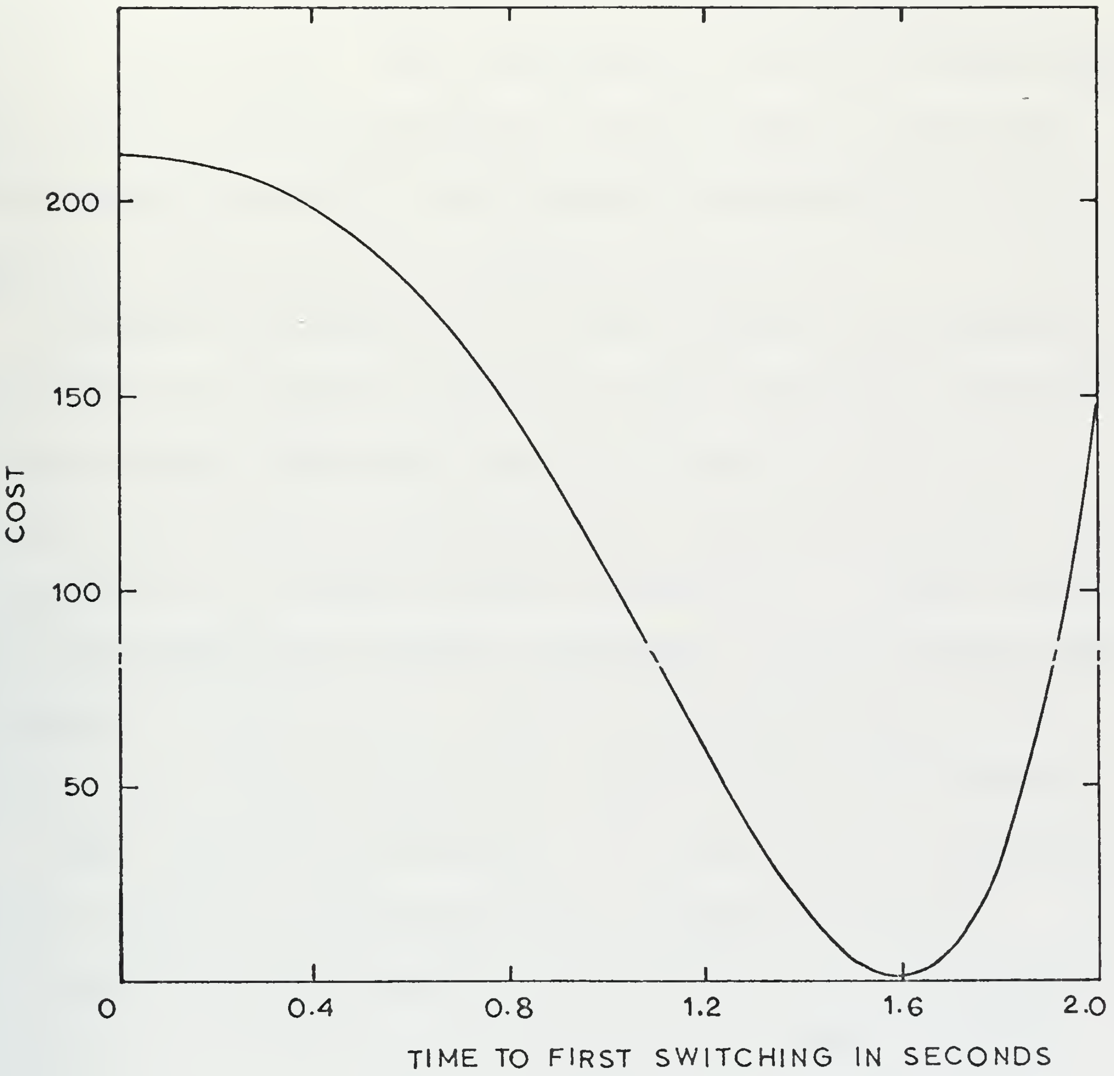


Figure 25. Cost Function for the Third Order, Switched-Parameter System of Figure 20 as a Function of the Time to the First Switching for a 10 Volt Input Step, where the Square of Both the Deviation of the States from the Desired Endpoint Values and the Deviation of the States from the Desired Final State During the Second Subinterval are Counted as Cost.

it was for Figure 22, but is increased in value slightly to about 1.6 seconds.

Again, a function minimization was done on the Cost function. The "optimum" switching time so obtained was, $t_1 = 1.600$ seconds. This optimization was also accomplished for the same three amplitudes of input step as were used in Figures 23 and 24. The "optimum" trajectories, $x_2(t)$ and $x_3(t)$, are plotted in Figures 26 and 27.

The main difference to be noted between the responses of Figures 23 and 26 is that the first "optimum" trajectory shows a slight undershoot, and the second a slight overshoot. Both optimum systems show a marked improvement over that shown by the same system in Figure 21. As was the case for the first Cost function considered for this system, the optimum switching time is seen to be independent of the input step function amplitude, and for the same reasons.

Cost function minimization is, thus, seen to provide a useful means of "optimizing" this particular kind of switched-parameter system. The utility of this kind of procedure for improving the responses of other kinds of switched parameter systems remains to be demonstrated.

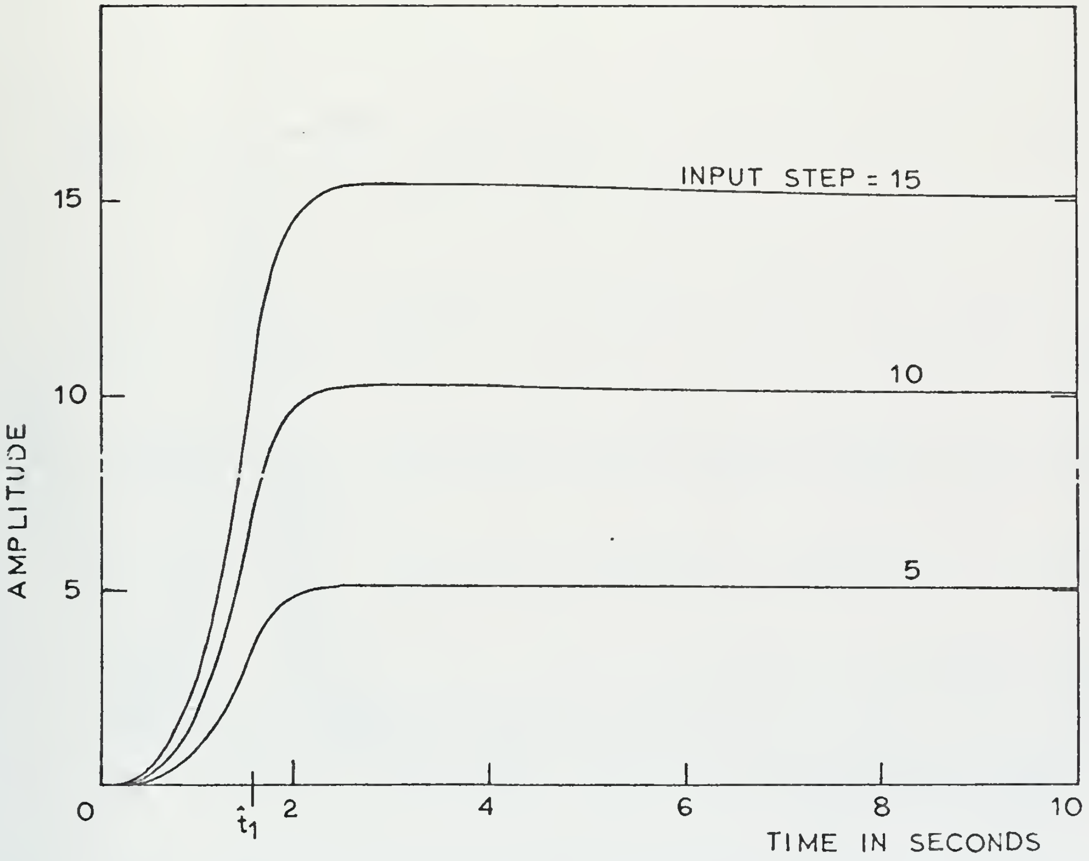


Figure 26. Optimum Time Response, $\hat{x}_2(t)$, of the Third Order, Switched-Parameter System of Figure 20 Obtained by Minimizing the Cost Function of Figure 25 with Respect to the Time to the First Switching, for Various Amplitude Input Steps. Optimum Switching Time, $\hat{t}_1$ = 1.600 Seconds.

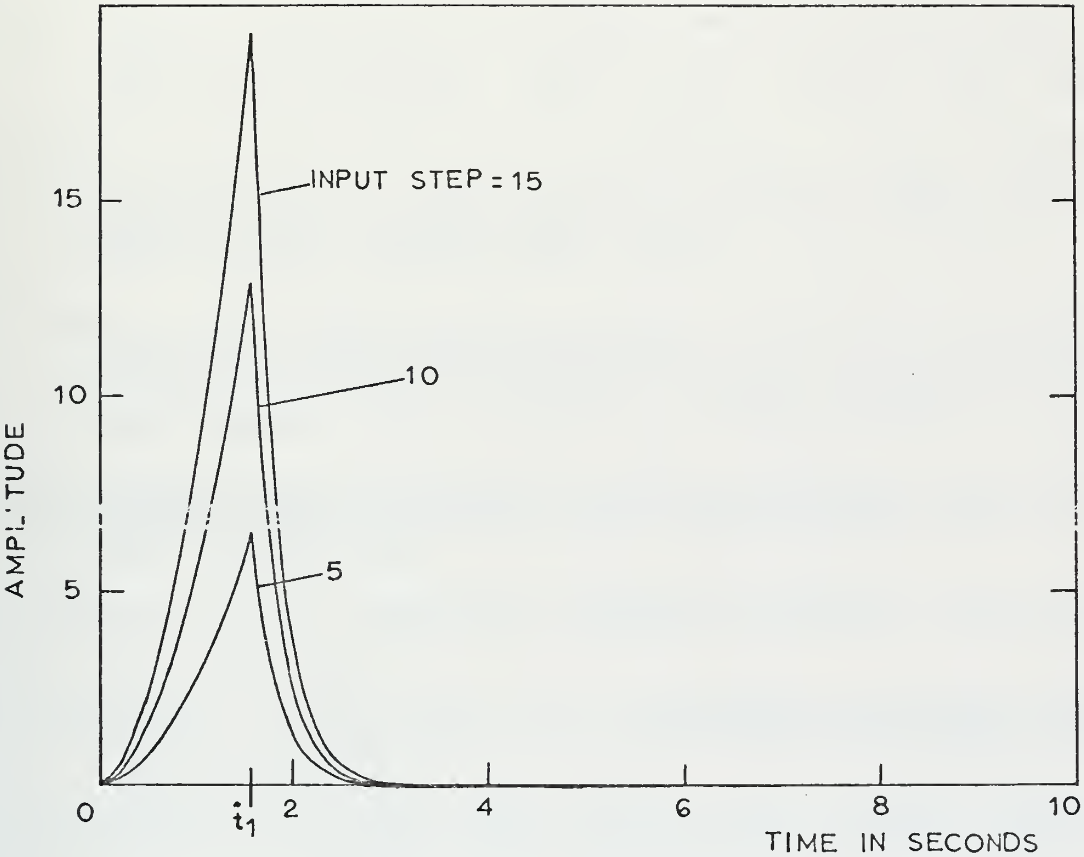


Figure 27. Optimum Time Response, $\hat{x}_3(t)$, for the Same System as in Figure 26.

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3. ABSTRACT

A general description of linear, switched-parameter systems is developed in terms of the mathematics of state variables. The restrictions imposed by the grouping of state variables and by the "core" states of the system upon the coefficient matrices are considered. Two procedures for determining the element values of those matrices are each illustrated with an example. A general expression for a Cost function to be used to measure system quality is developed and illustrated with two examples. Extensive recommendations for future work are made. Several examples of the utility of a Cost function minimization technique for the improvement of the step responses of some switched-parameter systems are presented.

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